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GIVEN BY
Hon. Chas. F. Adams

THE
ARITHMETIC
OF
INFINITES,
AND THE
DIFFERENTIAL METHOD;
ILLUSTRATED BY
EXAMPLES.

Hæ tibi erunt Artes. — VIRG.

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THE

PREFACE.

THE method which the antients generally made uſe of for finding the areas of figures, or the ſolidities of bodies, was by the help of other inſcribed and circumſcribed figures; after ſuch a manner, that the inſcribed figures being continually divided into very ſmall parts, the ſum of the inſcribed parts, approached ſo near the true figure, as to fall ſhort of it by a quantity leſs than any given quantity. And in like manner the circumſcribed figure was divided into ſo many ſmall parts, that their ſum exceeded the given figure, by a quantity alſo leſs than any given one. So they always had the value of the figure propoſed, between the two values of the inſcribed and circumſcribed figures; and by this method found out its content. But this way being for the moſt part ſo very laborious, that it put the ſucceeding geometers upon finding out ſhorter methods; and thus came up the Method of Indiviſibles, *or* the Arithmetic of Infinites. *This method ſolves theſe ſorts of Problems with admirable brevity, and extreme facility; and that by a help of a few Propoſitions, which reduces the whole buſineſs to the ſumming up the powers of an arithmetical progreſſion, conſiſting of an infinite number of terms. As we ſhall have frequent occaſion for this afterwards; I have here delivered that me-*

thod at large, with many examples to ſhew the uſe of it. The whole is contained in Sect. I. *of the following Book.*

I have likewiſe given the principles of the differential method, *and the* Interpolation of Series, *in* Sect. II. *an art extremely uſeful in almoſt all parts of the Mathematics, eſpecially in Aſtronomy. For it is frequently required to find the poſition of a planet or comet at ſome given time when an obſervation cannot be made; but by help of other obſervations, we can find it to that time, by the interpolation of ſeries. This method I have alſo made very clear and plain, and illuſtrated the ſame with variety of examples.*

W. Emerſon.

SECT. I.

The Arithmetic of Infinites.

THE *Arithmetic of Infinites* is the art of ſumming up the powers of a ſeries of quantities in arithmetical progreſſion; whoſe number of terms is infinite, and their common difference infinitely ſmall.

It is alſo called the *Method of Indiviſibles*, becauſe magnitudes are here ſuppoſed to be reſolved into their indiviſible parts, or ſo far as there is any occaſion to conſider them as ſuch.

PROP. I.

If a ſeries of equal quantities be added together, the ſum of all the terms is equal to any one of them multiplied into the number of terms.

For $a+a+a\ldots$ to n terms is $= n\times a$.

PROP. II.

If there be a ſeries of quantities in arithmetical progreſſion, and continued ad infinitum; as 0, a, $2a$, $3a$, $4a\ldots$ *to* na; *the ſum of all the terms is* $=\frac{1}{2}na$, *or to half ſo many times the greateſt.*

For by (Prop. 7. Arith. Progreſſion) the ſum $=\frac{0+na}{2}\times\overline{n+1}$, or $\frac{na\times\overline{n+1}}{2}$, but ſince n is infinite, $n+1=n$. therefore the ſum $=\frac{n}{2}\times na=\frac{n}{2}\times$ laſt term.

Cor. *The greateſt term multiplied by the number of terms, is equal to twice the ſum of the ſeries.*

PROP. III.

If there be a ſeries of ſquares, whoſe roots are in arithmetical progreſſion beginning at o, and continued ad infinitum, as $0,\ a^2,\ \overline{2a}^2,\ \overline{3a}^2,\ \overline{4a}^2 \cdots$ to $\overline{na}^2$; *the ſum of all the terms is* $= \frac{1}{3}n\times\overline{na}^2$, *or* $\frac{1}{3}$ *ſo many times the greateſt.*

For let $s = 0+a^2+\overline{2a}^2+\overline{3a}^2 \ldots$ to $\overline{na}^2$, L = laſt or greateſt term. Then by the *Method of Increments*, $\underset{\cdot}{s}=\overline{\overline{n+1.a}}^2 = z^2$, putting $\overline{n+1}.a=z$. But (by Cor. Ex. 10. Prop. XIII. *Increments)*, the integral of $x^n = \frac{x^{n+1}}{\overline{n+1}.\underset{\cdot}{x}} - \frac{x^n}{2}$ &c. Or writing z for x, and 2 for n, the int. of $z^2 = \frac{z^3}{3\underset{\cdot}{z}} - \frac{z^2}{z}$ &c. But ſince n is infinite, z is infinite, and z^3 is infinitely greater than z^2; whence int. of z^2, that is $s = \frac{z^3}{3\underset{\cdot}{z}}$. But $\underset{\cdot}{n}=1$, and $\underset{\cdot}{z} = \underset{\cdot}{n}a = a$; therefore $s = \frac{z^3}{3a} = \frac{\overline{\overline{n+1.a}}^3}{3a} = \frac{\overline{n+1.a}}{3a}\times\overline{\overline{n+1.a}}^2$. But as n is infinite, $n+1=n$; and $s = \frac{na}{3a}\times\overline{na}^2 = \frac{1}{3}n\times\overline{na}^2 = \frac{1}{3}\times n\text{L}$.

Cor. *The greateſt term multiplied by the number of terms, is equal to thrice the ſum of the ſeries.*

PROP. IV.

In a ſeries of cubes, whoſe roots are in arithmetical progreſſion, beginning at o, and infinitely continued; *as* $0,\ a^3,\ \overline{2a}^3,\ \overline{3a}^3, \ldots\ldots$ *to* $\overline{na}^3$; *the ſum* *of all the terms is* $= \frac{1}{4}n\times\overline{na}^3$, *or* $\frac{1}{4}$ *ſo many times the greateſt term.*

For put $s = o + a^3 + \overline{2a}^3 + \overline{3a}^3 \ldots + \overline{na}^3$, L $=$ greateſt term, then by the method of increments $s = \overline{\overline{n+1.a}}^3 = z^3$, putting $\overline{n+1}.a = z$. But (by Cor. Ex. 10. Prop. XIII. *Increments*). The integral of $x^n = \frac{x^{n+1}}{\overline{n+1}.\underset{\cdot}{x}} - \frac{x^n}{2}$ &c. or putting z for x, and 3 for n; the int. of $z^3 = \frac{z^4}{4\underset{\cdot}{z}} - \frac{z^3}{\underset{\cdot}{z}}$ &c. But $\underset{\cdot}{n} = 1$, and $\underset{\cdot}{z} = \underset{\cdot}{n}a = a$. Alſo ſince n is infinite, z is infinite, and z^4 is infinitely greater than z^3; and $n+1 = n$. Whence int. z^3 or $s = \frac{z^4}{4\underset{\cdot}{z}} = \frac{\overline{\overline{n+1.a}}^4}{4a} = \frac{\overline{na}^4}{4a} = \frac{\overline{na}}{4a} \times \overline{na}^3 = \frac{1}{4}n\times$L.

Cor. *The greateſt term multiplied by the number of terms, is equal to four times the ſum of the ſeries.*

PROP. V.

If there be a ſeries of mth powers, whoſe roots are in arithmetical progreſſion, beginning at o, and continued to infinity, as o, a^m, $\overline{2a}^m$, $\overline{3a}^m$, $\overline{4a}^m \ldots$ *to* $\overline{na}^m$, *the ſum of all the terms is* $= \frac{n}{m+1} \times \overline{na}^m = \frac{1}{m+1}$ $\times$ *ſo many times the greateſt term.*

Let $s = o + a^m + \overline{2a}^m + \overline{3a}^m \ldots + \overline{na}^m$, L $=$ laſt term; and by the method of *increments*, $s = \overline{\overline{n+1.a}}^m = z^m$. And (by the ſame Cor. Ex. 10. Prop. XIII. *Increments*) $s =$ int. $z^m = \frac{z^{m+1}}{\overline{m+1}\cdot\underset{\cdot}{z}} - \frac{z^m}{2}$ &c. But

$n = 1$, and $z = na = a$; and ſince n is infinite, $\overline{n+1}.a = na$, and z^m vaniſhes, being infinitely leſs than z^{m+1}; therefore $s = \frac{z^{m+1}}{m+1.a} = \frac{\overline{\overline{n+1}.a}^{m+1}}{m+1.a} = \frac{\overline{na}^{m+1}}{m+1.a} = \frac{na}{m+1.a} \times \overline{na}^{m} = \frac{n}{m+1} \times \overline{na}^{m} = \frac{1}{m+1} \times nL$.

Cor. *The greateſt term multiplied by the number of terms, is equal to* $m+1$ *times the ſum of the ſeries.*

PROP. VI.

To reſolve a Problem by the Arithmetic of Infinites.

The chief uſe of this method is to find the content of geometrical quantities, as of ſuperficies and ſolids, by reducing them to their indiviſible parts, or firſt elements; and then finding the ſum of all theſe, as directed by the following

RULE.

Suppoſe a line to be compoſed of an infinite number of points; a ſurface of an infinite number of lines, drawn parallel and equidiſtant from one another; and a ſolid of an infinite number of planes, parallel and equidiſtant; then obſerve what ſort of a ſeries it is that makes up that quantity. Find the ſum of that ſeries by ſome of the foregoing propoſitions, which will be the content required.

SCHOLIUM.

Here lines are ſuppoſed to be made up of an infinite number of equidiſtant points; plain figures of an infinite number of parallel lines, like the threads in a piece of cloth. And ſolids of an infinite

finite number of planes, like the leaves in a book. Fig. Yet theſe points, lines and ſurfaces, are not really ſuch, but are called ſo by reaſon of the ſimilitude. For theſe points are in ſtrictneſs, infinitely ſhort lines; the lines, infinitely narrow parallelograms, and the planes that compoſe the ſolids, infinitely thin ſolids; ſo that whenever we ſpeak of points, lines, or planes, compoſing any geometrical magnitude, it muſt always be underſtood in the ſenſe here mentioned. Moreover, if all theſe parallel lines that make up a ſurface be perpendicular to the axis of a curve; then theſe parallel lines will cut the curve, or any other line, at oblique angles, and the points ſuppoſed to conſtitute that curve, or that line, will be greater than the correſpondent points in the axis, in ſome certain proportion; which proportion is to be inveſtigated by the property of the figure.

Ex. 1.

To find the area of the Parallelogram ABCD. 1.

Draw AP perp. to the baſe CD; divide AP into an infinite number of equal parts, thro' all which ſuppoſe ſo many parallels drawn to the baſe CD. Then all theſe equal lines compoſe the area of the parallelogram, and their number is denoted by AP. Then by Prop. I. the ſum of all is = one of them multiplied by the number of all; that is = CD×AP, or the baſe multiplied into the height.

Ex. 2.

To find the content of a cylinder ABCD. 2.

Draw AP perp. to the baſe, which divide into an infinite number of equal parts; thro' which draw as many planes parallel to the baſe CD. Then all theſe equal planes compoſe the ſolidity of the cylinder; whoſe number is AP. Whence by Prop. I. the ſum of all is = one CD × their number AP; that

Fig. that is, the cylinder $=$ CD×AP, or the baſe multiplied by the height.

And the ſame way the ſolidity of a parallelopipedon or of a priſm is calculated.

Ex. 3.

3. *To find the area of a triangle* ABC.

Draw AP perp. to the baſe BC; divide AP into an infinite number of equal parts; thro' which ſuppoſe lines drawn parallel to BC. Then theſe lines make up the triangle, and are a ſeries of terms in arithmetical progreſſion beginning at *o*, and ending at BC; and the number of them is denoted by AP. Then (by Prop. II.) the ſum of all the terms is $= \frac{1}{2}AP \times BC$; that is, the area of the triangle $= \frac{AP \times BC}{2}$.

Ex. 4.

4. *To find the ſolidity of a cone* ABC.

Draw AP perp. to the baſe BC; divide AP into an infinite number of equal parts; thro' which draw planes parallel to the baſe BC. Then theſe planes make up the cone; and their diameters are a ſeries of quantities in arithmetical progreſſion, beginning at *o*, and ending at BC; therefore the planes or circles themſelves are proportional to a ſeries of ſquares, whoſe roots are in arithmetical progreſſion. Whence (by Prop. III.) the ſum of all the ſeries is $= \frac{1}{3}AP \times$ baſe BC; that is, the cone is $= \frac{1}{3}$ height × baſe.

And the ſame way it is proved, that a pyramid is $\frac{1}{3}$ the height × by the baſe.

Ex. 5.

To find the ſurface of the right cone ABC. Fig. 5.

Draw the axis AP, and DF perp. to it, and put $AP=h$, $BP=b$, $AB=s$, $AD=x$, $c=3.1416$; then by ſimilar triangles, AP (h) : PB (b) : : AD (x) : $DE=\frac{bx}{h}$; and $\frac{2cbx}{h}=$ periphery at F. Then the ſum of all the peripheries at F makes up the ſurface of the cone. But the points in AB being greater than thoſe in AP, in the ratio of AB to AP. Therefore the periphery at F muſt be increaſed to $\frac{2scbx}{hh}$ or $\frac{2scb}{hh}\times x$, and then the ſum of all theſe peripheries make up the ſurface. But the ſum of all the xs is 0, 1, 2, 3, &c. to AP; that is (by Prop. II.) $\frac{1}{2}AP\times AP$ or $\frac{1}{2}AP^2$. Therefore the conic ſurface is $=\frac{2scb}{hh}\times\frac{1}{2}hh=bcs=$ circumference of $BC\times AB$.

Otherwiſe.

Put $BP=b$, $AB=s$, $AF=z$, then by ſimilar triangles AB (s) : BP (b) : : AF (z) : $FD=\frac{bz}{s}$; then $\frac{2cbz}{s}=$ periphery at F; and the ſum of all theſe peripheries make up the ſurface. Now all the z's conſtitute a ſeries of terms in arithmetical progreſſion, whoſe laſt term, and alſo the number of terms is s: and therefore the ſum (by Prop. II.) is $\frac{1}{2}ss$. Therefore the ſum of all the peripheries, or the ſurface of the cone is $=\frac{2cb}{s}\times\frac{1}{2}ss=cbs$ as before.

Fig.

Ex. 6.

6. *To find the ſolidity of any ſegment of a ſphere* ADF.

Draw the diameter AB, and DEF perp. to it; and put radius AC $= r$, AE $= x$, DE $= y$, $c = 3.1416$; then by the nature of the circle $yy = 2rx - xx$. Then if AE be divided into an infinite number of equal parts, and planes be drawn thro' them parallel to DF; the ſum of all theſe circular planes make up the ſegment. Now one of theſe circles is cyy or $2crx - cxx$; and all the x's conſtitute a ſeries of terms in arithmetic progreſſion, whoſe ſum (by Prop. II.) is $\frac{1}{2}xx$. Alſo the ſum of all the xx's (by Prop. III.) is $\frac{1}{3}x^3$, where the greateſt term is xx, and number of terms, x. Therefore the ſum of all the circles conſtituting the ſegment, is $2cr \times \frac{1}{2}xx - c \times \frac{1}{3}x^3$, or $crxx - \frac{cx^3}{3}$. Therefore the ſegment DAF $= \overline{crx - \frac{1}{3}cxx} \times x$: that is, the ſegment DAF is equal to a cylinder whoſe height is AE, and baſe $crx - \frac{1}{3}cxx$, or $\frac{1}{2}cyy + \frac{1}{6}cxx = \frac{1}{2}$ circle DF $+ \frac{1}{6}$ circle AE.

Cor. *The hemiſphere, as likewiſe the ſphere is* $\frac{2}{3}$ *the circumſcribing cylinder.*

For when $x = r$, the ſegment becomes $\overline{crr - \frac{1}{3}crr} \times r = \frac{2}{3}crr \times r$.

Ex. 7.

To find the ſurface of the ſegment of a ſphere DAF.

Let CA $= r$, AE $= x$, DE $= y$, $c = 3.1416$; then the circumference at D $= 2cy$. And the ſum of all the peripheries make up the ſurface of the ſegment DAF. But the points in AD being greater than thoſe in AE, in proportion of the tangent TD to TE, or of DC to DE; therefore $y : r :: 1 : \frac{r}{y}$ $=$ magnitude of a point at D. Whence $2cy \times \frac{r}{y}$ or $2cr$ is the true periphery at D. And the ſum of all theſe peripheries $=$ ſurface of the ſegment; and the number of them is denoted by AE. Therefore (by Prop. I.) that ſum $= 2cr \times \mathrm{AE}$; that is, the ſurface of the ſegment DAF $= 2crx$.

Cor. *The ſurface of the hemiſphere is* $= 2crr$; *and of the whole ſphere* $4crr$, *or four great circles of the ſphere.*

⁂

SECT. II.

The Differential Method.

THE *Differential Method*, is the art of working with the differences of quantities. By this method any term of a ſeries may be found from the ſeveral orders of differences being given; or *vice verſa*, any difference may be found from having the terms of the ſeries given: it likewiſe ſhews how to find the ſum of ſuch a ſeries. And it gives rules to find by interpolation, any intermediate term, which is not expreſſed in the ſeries, by having its place or poſition given.

When any ſeries of quantities is propoſed, take the firſt term from the ſecond, the ſecond from the third, the third from the fourth, &c. then all theſe remainders make a new ſeries, called the *firſt order of differences*. In this new ſeries take the firſt term from the ſecond, the ſecond from the third, the third from the fourth, &c. as before; and theſe remainders make another ſeries, called the *ſecond order of differences*. In like manner, in this ſeries, take the firſt term from the ſecond, the ſecond from the third, &c. And theſe will make a ſeries, called the *third order of differences*. And after this manner you may proceed as far as you will. Thus in the following propoſition, A, b, c, d, e, &c. is the ſeries; B, B^2, B^3, B^4, &c. the firſt order of differences; C, C^2, C^3, &c. the ſecond order of differences; D, D^2, &c. the third order; E, &c. the fourth order, and ſo on. But the firſt terms of theſe ſeveral orders of differences, as B, C, D, E, &c. are thoſe that are principally made uſe of in calculations by this method.

PROP.

PROP. I.

If there be any ſeries A, *b*, *c*, *d*, *e*, *&c. and if there be taken, the firſt differences* B, B^2, B^3, *&c. the ſecond differences* C, C^2, C^3, *&c. the third differences* D, D^2, D^3, *&c. and ſo on.*

Then if T *ſtand for the firſt term of the nth differences; I ſay,* $\pm T = A - nb + n \times \frac{n-1}{2} c - n \times \frac{n-1}{2} \times \frac{n-2}{3} d + n \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} e -$ *&c. that is,* $+T$, *when n is even, and* $-T$ *when n is odd.*

The ſeveral Orders of differences being taken as before directed, will ſtand thus. Then

$$\begin{array}{ccccc}
A & & & & \\
 & B & & & \\
b & & C & & \\
 & B^2 & & D & \\
c & & C^2 & & E \\
 & B^3 & & D^2 & \\
d & & C^3 & & \\
 & B^4 & & \&c. & \\
e & & & &
\end{array}$$

ſeries A, b, c, d, e, &c.
1ſt diff. $b-A$, $c-b$, $d-c$, $e-d$, &c.
2d diff. $c-2b+A$, $d-2c+b$, $e-2d+c$, &c.
3d diff. $d-3c+3b-A$, $e-3d+3c-b$, &c.
4th diff. $e-4d+6c-4b+A$, &c.

That is, $B = b - A$, $C = c - 2b + A$, $D = d - 3c + 3b - A$, $E = e - 4d + 6c - 4b + A$, &c. or $-B = A - b$, $+C = A - 2b + c$, $-D = A - 3b + 3c - d$, $+E = A - 4b + 6c - 4d + e$, &c. where putting T ſucceſſively equal to B, C, D, E, &c. and $n = 1, 2, 3, 4$, &c. the Prop. will be evident.

Cor. *Hence* $A = A$, *the firſt term.*
$B = -A + b$, *the firſt difference.*
$C = A - 2b + c$, *the 2d difference.*
$D = -A + 3b - 3c + d$, *the 3d difference.*
$E = A - 4b + 6c - 4d + e$, *the 4th difference.*
$F = -A + 5b - 10c + 10d - 5e + f$, *the 5th difference.*
&c.

PROP.

PROP. II.

If A, *b*, *c*, *d*, *e*, *&c. be any ſeries; and there be taken* B, C, D, E, *&c. the firſt of the ſeveral orders of differences.*

Then I ſay the nth term of the ſeries, will be $= A + \frac{n-1}{1}B + \frac{n-1}{1} \times \frac{n-2}{2}C + \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3}D + \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4}E + \&c.$

For from the equations in the laſt Prop. viz. $B = b - A$, $C = c - 2b + A$, &c. we have by tranſpoſing,
$b = A + B$, $= -A + 2b + C = -A + 2A + 2B + C$ (expunging b); that is,
$c = A + 2B + C$, $d = A - 3b + 3c + D = A - 3A - 3B + 3A + 6B + 3C + D$ (expunging b and c); that is, $d = A + 3B + 3C + D$. Alſo $e = -A + 4b - 6c + 4d + E =$ (expunging b, c, d) $-A + 4A + 4B - 6A - 12B - 6C + 4A + 12B + 12C + 4D + E$; that is, $e = A + 4B + 6C + 4D + E$, &c.

Then puting A, b, c, d, &c. for the nth term; and n ſucceſſively $= 1, 2, 3, 4$, &c. the ſeries will be evident.

Cor. 1. *If* d', d'', d''', *&c. be the firſt of the firſt, ſecond, third order, &c. of differences; then*

The nth term of the ſeries, A, *b*, *c*, *d*, *&c. will be* $= A + \frac{n-1}{1}d' + \frac{n-1}{1} \times \frac{n-2}{2}d'' + \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3}d''' + \frac{n-1}{1} \times \frac{n-2}{2} \times \frac{n-3}{3} \times \frac{n-4}{4}d'''' +$ &c.

For $B = d'$, $C = d''$, $D = d'''$, &c. And the coefficients are the unciæ of the $\overline{n-1}^{th}$ power.

Cor. 2. *Hence alſo it follows, that any term of a given ſeries may be accurately determined; if the differences of any order happen at laſt to be equal.*

Cor. 3. *Hence* $A=A$, *the firſt term.*
$b=A+B$, *the 2d term.*
$c=A+2B+C$. *the 3d term.*
$d=A+3B+3C+D$, *the 4th term.*
$e=A+4B+6C+4D+E$, *the 5th term.*
$f=A+5B+10C+10D+5E+F$, *the 6th term.*
$g=A+6B+15C+20D+15E+6F+G$, *the 7th term.*
&c.

PROP. III.

If a, b, c, d, e, *&c. be any ſeries; and* d', d'', d''', *&c. the firſt of the ſeveral orders of differences; Then*

The ſum of n *terms of the ſeries, is* $=na+n\times\frac{n-1}{2}d'+n\times\frac{n-1}{2}\times\frac{n-2}{3}d''+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}d'''+n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}\times\frac{n-4}{5}d''''$ + &c.

For in the ſeries of quantities,

$o\ ,\ a\ ,\ a+b\ ,\ a+b+c\ ,\ a+b+c+d$, &c.
1ſt diff. are $a\ ,\ b\ ,\ c\ ,\ d$, &c.
2d diff. $d'\ ,\ d'2\ ,\ d'3$, &c.
3d diff. $d''\ ,\ d''2$, &c.
4th diff. d''' , &c.

Therefore (by Cor. 1. Prop. II.) the $\overline{n+1}^{th}$ term of the ſeries, o, a, $a+b$, $a+b+c$, $a+b+c+d$, &c. or the $\overline{n}^{th}$ term of the ſeries, a, $a+b$, $a+b+c$, $a+b+c+d$, &c. is $=o+na+n\times\frac{n-1}{2}d'+n\times\frac{n-1}{2}\times\frac{n-2}{3}d''$ + &c. But the nth term of the ſeries a, $a+b$, $a+b+c$, &c. is the ſum of n terms of the ſeries, a, b, c, d, &c. and therefore equal to $na+n\times\frac{n-1}{2}d'+n\times\frac{n-1}{2}\times\frac{n-2}{3}d''$ + &c.

PROP.

PROP. IV.

If there be ſeveral ſets of quantities in arithmetic progreſſion, whoſe number is m, and the common differences p, q, r, s, &c. and if a new ſeries be formed by multiplying the reſpective terms of all theſe ſeries into one another; then I ſay the mth order of differences will be equal to one another, and to pqrs, &c. $\times$ *into* $1\times2\times3\times4$, *&c. to m.*

1. *Suppoſe two ſeries;*

as a, $a+p$, $a+2p$, $a+3p$, $a+4p$, &c.
and b, $b+q$, $b+2q$, $b+3q$, $b+4q$, &c.

then by multiplying we have ab, $\overline{a+p}\times\overline{b+q}$, $\overline{a+2p}\times\overline{b+2q}$, &c. that is,

ſeries	1 Diff.	2 Diff.
ab		
$ab+pb+aq+pq$	$pb+aq+pq$	
$ab+2pb+2aq+4pq$	$pb+aq+3pq$	$2pq$
$ab+3pb+3aq+9pq$	$pb+aq+5pq$	$2pq$
&c.		

therefore the 2d differences are equal, and equal to $pq\times1\times2$, where $m=2$.

2. *Suppoſe three ſeries.*

as a, $a+p$, $a+2p$, $a+3p$, $a+4p$, &c.
b, $b+q$, $b+2q$, $b+3q$, $b+4q$, &c.
c, $c+r$, $c+2r$, $c+3r$, $c+4r$, &c.

The terms of theſe being multiplied, produce a ſeries whoſe terms are,

abc
$\overline{a+p}\times\overline{b+q}\times\overline{c+r}$
$\overline{a+2p}\times\overline{b+2q}\times\overline{c+2r}$
$\overline{a+3p}\times\overline{b+3q}\times\overline{c+3r}$
$\overline{a+4p}\times\overline{b+4q}\times\overline{c+4r}$
&c.

theſe being multiplied at length, and ſubtracted from one another, will give the following ſeries, which is the firſt order of differences,

$$abr+bcp+aqc+pqc+aqr+pbr+pqr$$
$$abr+bcp+aqc+3pqc+3aqr+3pbr+7pqr$$
$$abr+bcp+aqc+5pqc+5aqr+5pbr+19pqr$$
$$abr+bcp+aqc+7pqc+7aqr+7pbr+37pqr$$

&c. And the differences of theſe, or the ſecond differences being taken, we have

$$2pqc+2aqr+2pbr+6pqr$$
$$2pqc+2aqr+2pbr+12pqr$$
$$2pqc+2aqr+2pbr+18pqr$$

and the third difference will be

$$6pqr$$
$$6pqr$$

here $m=3$, and the mth difference is $pqr\times1\times2\times3$.

3. *Suppoſe four ſeries.*

As a, $a+p$, $a+2p$, $a+3p$, $a+4p$, &c.
b, $b+q$, $b+2q$, $b+3q$, $b+4q$, &c.
c, $c+r$, $c+2r$, $c+3r$, $c+4r$, &c.
d, $d+s$, $d+2s$, $d+3s$, $d+4s$, &c.

and when multiplied, produce

$$abcd$$
$$+\overline{a+p}\times\overline{b+q}\times\overline{c+r}\times\overline{d+s}$$
$$+\overline{a+2p}\times\overline{b+2q}\times\overline{c+2r}\times\overline{d+2s}$$
$$+\overline{a+3p}\times\overline{b+3q}\times\overline{c+3r}\times\overline{d+3s}$$
$$+\overline{a+4p}\times\overline{b+4q}\times\overline{c+4r}\times\overline{d+4s}$$
$$+\overline{a+5p}\times\overline{b+5q}\times\overline{c+5r}\times\overline{d+5s}$$

&c.

Let ABCD be any term;

and the next, $\overline{A+p}\times\overline{B+q}\times\overline{C+r}\times\overline{D+s}$; that is, $\overline{AB+Aq+pB+pq}\times\overline{CD+Cs+rD+rs}=ABCD+AB\times\overline{Cs+rD+rs}+CD\times\overline{Aq+pB+pq}+\overline{Aq+pB+pq}\times\overline{Cs+rD+rs}$. From this, ſubtract ABCD, and

the remainder is the firſt difference $=AB\times\overline{Cs+rD+rs}+CD\times\overline{Aq+pB+pq}+\overline{Aq+pB+pq}\times\overline{Cs+rD+rs}$. To abridge the work, keep in only ſuch terms as involve the moſt of the quantities ABCD, for the reſt will vaniſh before you come to the laſt differences. By this means the firſt difference becomes only $AB\times\overline{Cs+rD}+CD\times\overline{Aq+Bp}$; that is, the firſt difference (1) $=ABCs+ABDr+CDAq+CDBp$; then the next term is evidently $\overline{A+p}\times\overline{B+q}\times\overline{C+r}\times s+\overline{A+p}\times\overline{B+q}\times\overline{D+s}\times r+\overline{A+p}\times\overline{C+r}\times\overline{D+s}\times q+\overline{B+q}\times\overline{C+r}\times\overline{D+s}\times p=\overline{AB+Aq+pB}\times\overline{C+r}\times s+\overline{AB+Aq+pB}\times\overline{D+s}\times r+\overline{AC+Ar+pC}\times\overline{D+s}\times q+\overline{BC+Br+qC}\times\overline{D+s}\times p=$ (2) $ABCs+ABrs+\overline{Aq+pB}\times\overline{C+r}\times s+ABDr+ABsr+\overline{Aq+pB}\times\overline{D+s}\times r+ACDq+ACsq+\overline{Ar+pC}\times\overline{D+s}\times q+BCDp+BCsp+\overline{Br+qC}\times\overline{D+s}\times p$; from this ſubtracting the equation (1), and the ſecond difference is $ABrs+\overline{Aq+pB}\times\overline{C+r}\times s+ABsr+\overline{Aq+pB}\times\overline{D+s}\times r+ACsq\times\overline{Ar+pC}\times\overline{D+s}\times q+BCsp+\overline{Br+qC}\times\overline{D+s}\times p=ABrs+ACqs+BCps,\ +ABrs+ADqr+BDpr,\ +ACsq+ADrq+CDpq,\ +BCsp+BDrp+CDpq$, &c. by abridging $=$ (3) $2ABrs+2ACqs+2BCps+2ADqr+2BDpr+2CDpq$, this is the 2d difference; and the next term is $2rs\times\overline{A+p}\times\overline{B+q}+2qs\times\overline{A+p}\times\overline{C+r}+2ps\times\overline{B+q}\times\overline{C+r}+2qr\times\overline{A+p}\times\overline{D+s}+2pr\times\overline{B+q}\times\overline{D+s}+2pq\times\overline{C+r}\times\overline{D+s}=$ (4) $2rs\times AB+2rs\times\overline{Aq+pB}+2qsAC+2qs\times\overline{Ar+pC}+2psBC+2ps\times\overline{Br+qC}+2qrAD+2qr\times\overline{As+pD}+2prBD+2pr\times\overline{Bs+qD}+2pqCD+2pq\times\overline{Cs+rD}$, &c. by abridging; from this laſt equation taking equation (3), the remainder is the 3d difference $=2rs\times\overline{Aq+pB}+2qs\times\overline{Ar+pC}+2ps\times\overline{Br+qC}+2qr\times\overline{As+pD}+2pr\times\overline{Bs+qD}+2pq\times\overline{Cs+rD}=$ (5) $6qrsA+6prsB$
$+6pqsC$

$+6pqsC+6pqrD$. Then the next term is (6) $6qrs\times\overline{A+p}+6prs\times\overline{B+q}+6pqs\times\overline{C+r}+6pqr\times\overline{D+s}$; from this equation, take equation (5), and there remains $6pqrs+6pqrs+6pqrs+6pqrs=24pqrs$ the fourth and laſt difference, ſo the fourth difference is $=pqrs\times1\times2\times3\times4$; and ſo on for more progreſſions.

Cor. 1. *In the ſeries* $a^m+\overline{a+p}^m+\overline{a+2p}^m+\overline{a+3p}^m$, *&c. the laſt and equal differences will be* p^m $\times$ *into* $1\times2\times3\times4$, *&c. to m terms.*

For then p, q, r, s, &c. are all equal.

Cor. 2. *In any adfected equation of the mth power, putting ſucceſſively any numbers of a ſeries in arithmetical progreſſion, for the unknown quantity; the mth differences will be equal.*

For the lower powers all vaniſh before you come to the laſt difference; and therefore that difference depends entirely on the higheſt power in the equation, which is the mth power.

PROP. V.

Let A, b, c, d, e, &c. be any ſeries; B, C, D, E, &c. the firſt of the ſeveral orders of differences; and let their poſition be ſuppoſed to be ſuch, that they may be at a units diſtance from one another; and let x be the diſtance of any term y (to be interpolated;) then I ſay,

$$y=A+xB+x\times\frac{x-1}{2}C+x\times\frac{x-1}{2}\times\frac{x-2}{3}D+x\times\frac{x-1}{2}\times\frac{x-2}{3}\times\frac{x-3}{4}E+\&c.$$

or $y=A+xB+\frac{x-1}{2}PC+\frac{x-2}{3}QD+\frac{x-3}{4}RE+\frac{x-4}{5}SF+\&c.$ where P, Q, R, &c. are the coefficients of B, C,D, &c.

For (Prop. II.) the *n*th term of the ſeries is $= A + \frac{n-1}{1}B + \frac{n-1}{1} \times \frac{n-2}{2}C$, &c. and the $\overline{n+1}^{th}$ term $= A + \frac{n}{1}B + n \times \frac{n-1}{2}C$, &c. which is the term at the *n*th diſtance from A; then put $x = n$, and the term at x diſtance from A, will be $P = A + xB + x \times \frac{x-1}{2}C + x \times \frac{x-1}{2} \times \frac{x-2}{3}D$, &c. And as this holds exactly, when P is any of the terms A, b, c, d, e, &c. Therefore it will be nearly true, when P is any intermediate term.

Cor. *Hence if ſeveral equidiſtant ordinates in any curve be given; the quantity of any other ordinate, whoſe poſition is given, may be found by this Prop. that is, ſuch an ordinate may be interpolated.*

PROP. VI. *Prob.*

If there be a ſeries of equidiſtant terms; a, b, c, d, e, &c. whoſe firſt differences are ſmall; to find any term wanting in that ſeries, having any number of terms given.

RULE.

Take the equation which ſtands againſt the number of given terms, in this table; and by reducing the equation, that term will be found.

1 $a-b=0$.
2 $a-2b+c=0$.
3 $a-3b+3c-d=0$.
4 $a-4b+6c-4d+e=0$.
5 $a-5b+10c-10d+5e-f=0$.
6 $a-6b+15c-20d+15e-6f+g=0$.
7 $a-7b+21c-35d+35e-21f+7g-h=0$.
8 $a-8b+28c-56d+70e-56f+28g-8h+i=0$.
n $a-nb+n \times \frac{n-1}{2}c-n \times \frac{n-1}{2} \times \frac{n-2}{3}d+\&c.=0$.

For (by Prop. I.) the quantities ſtanding againſt the numbers 1, 2, 3, 4, &c. are the firſt of the firſt, ſecond, third, fourth, &c. differences. Now as the firſt differences are ſmall, the other differences will decreaſe, and the laſt of them may be taken for nothing. And the more terms there are given, the nearer the equation approximates; likewiſe the quantity ſought will be determined more exactly, as it is nearer the middle of the equation; the error being almoſt reciprocally as the coefficient. Fig.

For Ex. Suppoſe b, c, d, e, f, g, are given to find a; againſt 6 you have $a-6b+15c-20d+15e-6f+g=0$, *which reduced gives* $a=6b-15c+20d-15e+6f-g$. *If the third* c *had been ſought, and the reſt given; then* $c=\frac{20d-15e+6f-g-a+6b}{15}$.

PROP. VII.

If a, b, c, d, e, &c. be any ſeries; and ſuppoſe the ſeveral terms placed at unequal diſtances from one another. To find any intermediate term, whoſe place or poſition among the reſt, is given. 8.

Suppoſe the ſeries a, b, c, d, e, &c. placed as ordinates in a curve; and let MN, MP, MQ, MR, MS, &c. be the correſpondent baſes, for which put n, P, Q, R, S, &c. reſpectively; alſo put $\mathrm{NP}=p$, $\mathrm{PQ}=q$, $\mathrm{QR}=r$, $\mathrm{RS}=s$, &c. and let x repreſent any baſe, and y its correſpondent term in the ſeries.

Then aſſume $y=\mathrm{A}+\mathrm{B}\times\overline{x-n}+\mathrm{C}\times\overline{x-n}\times\overline{x-\mathrm{P}}+\mathrm{D}\times\overline{x-n}\times\overline{x-\mathrm{P}}\times\overline{x-\mathrm{Q}}+\mathrm{E}\times\overline{x-n}\times\overline{x-\mathrm{P}}\times\overline{x-\mathrm{Q}}\times\overline{x-\mathrm{R}}$, &c. continued to as many terms as there are ordinates, or terms of the ſeries. Conſequently x will aſcend to a power, one leſs than the number of terms.

Fig. 8. Then put a, b, c, d, e, &c. ſucceſſively for y. and n, P, Q, R, S, &c. for x.

And theſe will produce as many equations as there are unknown quantities, A, B, C, D, E, &c. which therefore will be determined as follows.

$$a = A$$

$$b = A + B \times \overline{P - n}$$

$$c = A + B \times \overline{Q - n} + C \times \overline{Q - n} \times \overline{Q - P}$$

$$d = A + B \times \overline{R - n} + C \times \overline{R - n} \times \overline{R - P} + D \times \overline{R - n} \times \overline{R - P} \times \overline{R - Q}$$

$$e = A + B \times \overline{S - n} + C \times \overline{S - n} \times \overline{S - P} + D \times \overline{S - n} \times \overline{S - P} \times \overline{S - Q} + E \times \overline{S - n} \times \overline{S - P} \times \overline{S - Q} \times \overline{S - R},$$

&c. Then inſtead of P—n, Q—n, Q—P, &c. put their values p, $\overline{p+q}$, q, &c. and you'll have the following equations.

$$a = A$$

$$b = A + Bp$$

$$c = A + B \times \overline{p+q} + C \times \overline{p+q} \times q$$

$$d = A + B \times \overline{p+q+r} + C \times \overline{p+q+r} \times \overline{q+r} + D \times \overline{p+q+r} \times \overline{q+r} \times r$$

$$e = A + B \times \overline{p+q+r+s} + C \times \overline{p+q+r+s} \times \overline{q+r+s} + D \times \overline{p+q+r+s} \times \overline{q+r+s} \times \overline{r+s} + E \times \overline{p+q+r+s} \times \overline{q+r+s} \times \overline{r+s} \times s.$$

Then take each equation from the following one, and you'll have,

$$b - a = Bp$$

$$c - b = Bq + C \times \overline{p+q} \times q$$

$$d - c = Br + C \times \overline{p+2q+r} \times r + D \times \overline{p+q+r} \times \overline{q+r} \times r$$

e—

$$e-d = Bs+C\times\overline{p+2q+2r+s}\times s+D\times\overline{p+q+r}\times\overline{q+2r+s}\times s+\overline{q+r+s}\times\overline{r+s}\times s+E\times\overline{p+q+r+s}\times\overline{q+r+s}\times\overline{r+s}\times s.$$

Fig. 8.

And dividing by the reſpecting coefficients of B, and then

$$\frac{b-a}{p}=B.$$

$$\frac{c-b}{q}=B+C\times\overline{p+q}=B1.$$

$$\frac{d-c}{r}=B+C\times\overline{p+2q+r}+D\times\overline{p+q+r}\times\overline{q+r}=B2.$$

$$\frac{e-d}{s}=B+C\times\overline{p+2q+2r+s}+D\times\overline{p+q+r}\times\overline{q+2r+s}+\overline{q+r+s}\times\overline{r+s}+E\times\overline{p+q+r+s}\times\overline{q+r+s}\times\overline{r+s}=B3,$$

&c.

And again ſubſtracting each equation from the next,

$$B1-B=C\times\overline{p+q}.$$

$$B2-B1=C\times\overline{q+r}+D\times\overline{p+q+r}\times\overline{q+r}.$$

$$B3-B2=C\times\overline{r+s}+D\times\overline{p+2q+2r+s}\times\overline{r+s}+E\times\overline{p+q+r+s}\times\overline{q+r+s}\times\overline{r+s}.$$

&c.

And dividing by the coefficients of C,

$$\frac{B1-B}{p+q}=C.$$

$$\frac{B2-B1}{q+r}=C+D\times\overline{p+q+r}=C1.$$

$$\frac{B3-B2}{r+s}=C+D\times\overline{p+2q+2r+s}+E\times\overline{p+q+r+s}\times\overline{q+r+s}=C2.$$

&c.

Fig. And taking the differences of theſe equations,

8. $C1 - C = D \times \overline{p+q+r}.$

$C2 - C1 = D \times \overline{q+r+s} + E \times \overline{p+q+r+s} \times \overline{q+r+s}.$

&c.

And dividing by the coefficient of D,

$$\frac{C1 - C}{p+q+r} = D,$$

$$\frac{C2 - C1}{q+r+s} = D + E \times \overline{p+q+r+s} = D1,$$

&c. And taking the differences as before,

$$D1 - D = E \times \overline{p+q+r+s}.$$

and $\dfrac{D1 - D}{p+q+r+s} = E$, &c.

And thus you may proceed as far as you will; whence to find the intermediate term, whoſe baſe is x, we have this

RULE.

Let a, b, c, d, e, &c. be the ſeries, M any fixed point, and
for MN, NP, PQ, QR, RS, &c.
put n, p, q, r, s, &c. and

$$\begin{array}{llll}
\text{let } B = \dfrac{b-a}{p}, & C = \dfrac{B1 - B}{p+q}, & D = \dfrac{C1 - C}{p+q+r}, & E = \dfrac{D1 - D}{p+q+r+s}. \\
B1 = \dfrac{c-b}{q}, & C1 = \dfrac{B2 - B1}{q+r}, & D1 = \dfrac{C2 - C1}{q+r+s}, & \&c. \\
B2 = \dfrac{d-c}{r}, & C2 = \dfrac{B3 - B2}{r+s}, & \&c. & \\
B3 = \dfrac{e-d}{s} & \&c. & & \\
\&c. & & &
\end{array}$$

Then the term anſwering to the baſe x, is

$$y = A + B \times \overline{x-n} + C \times \overline{x-n} \times \overline{x-n+p} + D \times \overline{x-n} \times \overline{x-n+p} \times \overline{x-n+p+q} + E \times \overline{x-n} \times \overline{x-n+p} \times \overline{x-n+p+q} \times \overline{x-n+p+q+r}, + \&c.$$

where

Fig. 8. where M may be taken at pleaſure, and may be either without or among the ordinates. And here x is ſuppoſed to lie towards the right hand; but if it be taken to the left from M, it muſt be made negative. And if any of the terms of the ſeries a, b, c, &c. be negative, the ordinate repreſenting it, muſt lie below the baſe MS.

9. Cor. *Suppoſing* B, C, D, E, *&c. found as before, and let* M *fall upon* N *at the firſt ordinate, then will* $n = 0$, *and let* p, q, r, s, *&c. be the ſeveral diſtances of the ordinates from each other. Then putting* x *for any baſe reckoned from* M; *then will the correſponding term to be interpolated be,*

$$y = A + Bx + Cx \times \overline{x-p} + Dx \times \overline{x-p} \times \overline{x-p-q} + Ex \times \overline{x-p} \times \overline{x-p-q} \times \overline{x-p-q-r} + \&c.$$

continued to as many terms, as there are terms in the given ſeries.

SCHOLIUM.

They that would ſee other methods of interpolating the terms in a given ſeries, may conſult the Philoſophical *Tranſactions*, No. 362; or *Sterling*'s Summiation of ſeries, engliſhed by *Mr. Holliday.*

PROP. VIII.

To find a curve line of the parabolic kind, which ſhall paſs through any given number of points.

RULE.

10. Let the points be A, B, C, D, E, &c. and from them to any right line MT given in poſition, let fall as many perpendiculars, AM, BP, CQ, DR, ES, &c. Then to find other points through which the curve muſt paſs; aſſume any point X, and put x for the baſe MX, and y for the correſponding ordinate XY. Then if the diſtances of the ordinates AM, BP, CQ, &c. be all equal; find the

Fig. 10. the value of y by Prop. IV. But if their diſtances be unequal, find the value of y by Prop. VII. which gives the point Y, through which the curve is to paſs; and thus you may find as many points as you will; through all which draw the curve required.

Cor. *Hence, the interpolation of the terms of a ſeries comes to the ſame thing, as finding the ordinates in a parabolic curve of ſome degree or other. As ſuppoſe the equation to the parabolic curve be* $y = A + Bx + Cxx + Dx^3 + Ex^4$, *&c. where the powers of* x *muſt aſcend to as many dimenſions as there are ordinates wanting* 1. *Therefore if ſuch a parabola be deſcribed as above directed, whoſe ſeveral ordinates repreſent the particular terms of a given ſeries; then to find any other term of that ſeries, is no more than finding the correſpondent ordinate of the curve.*

PROP. IX.

To find the area of any given curve.

RULE.

11. Thro' any number of points in it, deſcribe a parabolic curve (by Prop. VIII.); and find its area, which will be nearly equal to the area of the curve propoſed.

Thus let ADF be the curve, and ſuppoſe any number of equidiſtant ordinates to be erected, as AM, BP, CQ, DR, &c. and thro' A, B, C, D, &c. ſuppoſe a parabolic curve to paſs, and let any baſe MX $= x$, and its ordinate XY $= y$; and the equation of the parabolic curve is $y = A + Bx + Cx^2 + Dx^3$ &c. continued to as many terms as there are ordinates. Then to find the area of this parabola; ſuppoſe x divided into an infinite number of equal parts, according to the Arithmetic of Infinites. Then (by Prop. I. Sect. I.) the ſum of all

Fig. 11.

all the A's is $= Ax$. And (Prop. II.) the ſum of all the Bx is $= \frac{Bxx}{2}$. And (Prop. III.) the ſum of all the $Cxx = \frac{Cx^3}{3}$. And (Prop. IV.) the ſum of all the $Dx^3 = \frac{Dx^4}{4}$, and ſo on. Therefore (by Prop. VII.) the ſum of all the y's, or of all the $A+Bx+Cx^2+Dx^3$, &c. that is, the area of the curve is $= Ax + \frac{Bx^2}{2} + \frac{Cx^3}{3} + \frac{Dx^4}{4}$, &c. $= \overline{A+\frac{1}{2}Bx+\frac{1}{3}Cx^2+\frac{1}{4}Dx^3}$ &c. $\times x$, or put L for x the length of the baſe; then the area $= \overline{A+\frac{1}{2}Bx+\frac{1}{3}Cx^2+\frac{1}{4}Dx^3}$, &c. $\times L$.

Now let a, b, c, d, e, &c. be the ordinates of the given curve placed at equal diſtances; and let MP, PQ, QR, &c. each $= 1$. Then from the nature of the parabolic curve, ($y = A+Bx+Cx^2$, &c.) putting $x = 0, 1, 2, 3, 4$, &c. and you'll have the following equations.

$$\begin{array}{l} a = A \\ b = A + B + C + D + E, \text{ \&c.} \\ c = A + 2B + 4C + 8D + 16E, \text{ \&c.} \\ d = A + 3B + 9C + 27D + 81E, \text{ \&c.} \\ e = A + 4B + 16C + 64D + 256E, \text{ \&c.} \\ \text{\&c.} \end{array}$$

Theſe are continued to as many terms as there are equations or ordinates. Then if the quantities A, B, C, D, &c. be expunged, by help of theſe equations, out of the area ($\overline{A+\frac{1}{2}Bx+\frac{1}{3}Cx^2}$, &c. $\times L$); you'll have the area expreſſed by the ordinates, a, b, c, d, &c.

1. For one ordinate.

Here $a = A$, $x = 0$, and $AL =$ area. Therefore $AL =$ area.

2. For two ordinates.

Here $x = 1$, $a = A$, $b = A + B$, therefore $B = b - a$, and the area $= \overline{A + \tfrac{1}{2}Bx} \times L = \overline{a + \frac{b-a}{2}} \times L$; that is, the area $= \frac{a+b}{2} L$.

3. For three ordinates.

Here $x = 2$, and

$$\begin{array}{l} a = A \\ b = A + B + C \\ c = A + 2B + 4C \\ \hline b - a = B + C \\ c - b = B + 3C \\ \hline c - 2b + a = 2C \\ \tfrac{1}{2}c - b + \tfrac{1}{2}a = C. \\ \hline B = b - a - C = -\tfrac{3}{2}a + 2b - \tfrac{1}{2}c \end{array}$$

Then the area $\overline{A + \tfrac{1}{2}B \times 2 + \tfrac{1}{3}C \times 4} \times L$ becomes

$$\left.\begin{array}{l} a \\ -\tfrac{3}{2}a + 2b - \tfrac{1}{2}c \\ +\tfrac{2}{3}a - \tfrac{4}{3}b + \tfrac{2}{3}c \end{array}\right\} \times L = \text{area.}$$

or $\overline{\tfrac{1}{6}a + \tfrac{2}{3}b + \tfrac{1}{6}c} \times L = \frac{a + c + 4b}{6} L$ the area.

4. For four ordinates.

Here $x = 3$, area $= A + \tfrac{3}{2}B + 3C + \tfrac{27}{4}D$.

and

$$\begin{array}{l} a = A \\ b = A + B + C + D \\ c = A + 2B + 4C + 8D \\ d = A + 3B + 9C + 27D \\ \hline b - a = B + C + D \\ c - b = B + 3C + 7D \\ d - c = B + 5C + 19D \\ \hline \end{array}$$

c—

$$\begin{array}{rcl} c-2b+a & = & 2C+6D \\ d-2c+b & = & 2C+12D \\ \hline d-3c+3b-a & = & 6D \end{array}$$

$$c-2b+a-d+3c-3b+a = 2C$$

or $2a-5b+4c-d = 2C$

and $a-\frac{5}{2}b+2c-\frac{1}{2}d = C$

$$-\tfrac{1}{6}a+\tfrac{1}{2}b-\tfrac{1}{2}c+\tfrac{1}{6}d = D$$

$$b-a-C-D = B = -\tfrac{11}{6}a+3b-\tfrac{3}{2}c+\tfrac{1}{3}d.$$

whence a

$$\left.\begin{array}{l} -\frac{11}{4}a+\frac{9}{2}b-\frac{9}{4}c+\frac{1}{2}d \\ +3a-\frac{15}{2}b+6c-\frac{3}{2}d \\ -\frac{9}{8}a+\frac{27}{8}b-\frac{27}{8}c+\frac{9}{8}d \end{array}\right\}\times L = \text{area}.$$

that is $\frac{a+d+3b+3c}{8}L = \text{area}.$

5. For five ordinates.

Here $x=4$, and the area $= A+2B+\frac{16}{3}C+16D+\frac{256}{5}E$.

$$\begin{array}{lrcrcrcrcrcr} \text{and} & a & = & A \\ & b & = & A & + & B & + & C & + & D & + & E \\ & c & = & A & + & 2B & + & 4C & + & 8D & + & 16E \\ & d & = & A & + & 3B & + & 9C & + & 27D & + & 81E \\ & e & = & A & + & 4B & + & 16C & + & 64D & + & 256E \end{array}$$

$$\begin{array}{rcrcrcrcr} \hline \text{then } b-a & = & B & + & C & + & D & + & E \\ c-2b+a & = & & & 2C & + & 6D & + & 14E \\ d-3c+3b-a & = & & & & & 6D & + & 36E \\ e-4d+6c-4b+a & = & & & & & & & 24E \\ \hline \end{array}$$

whence $E = \frac{1}{24}e-\frac{1}{6}d+\frac{1}{4}c-\frac{1}{6}b+\frac{1}{24}a.$

$$D = \frac{d-3c+3b-a-36E}{6} = -\tfrac{1}{4}e+\tfrac{7}{6}d-2c+\tfrac{3}{2}b-\tfrac{5}{12}a.$$

$$C = \frac{c-2b+a}{2}-3D-7E = \tfrac{11}{24}e-\tfrac{7}{3}d+\tfrac{19}{4}c-\tfrac{13}{3}b+\tfrac{35}{24}a.$$

$$B = b-a-C-D-E = -\tfrac{1}{4}E+\tfrac{4}{3}d-3c+4b-\tfrac{25}{12}a.$$

Theſe values of A, B, C, &c. ſubſtituted in the value of the area will give the area $= \frac{7}{90}e + \frac{16}{45}d + \frac{2}{15}c + \frac{16}{45}b + \frac{7}{90}a : \times L$; that is, the area $=$

$$\frac{7\times\overline{a+e}+32\times\overline{b+d}+12c}{90}.$$

6. If ſix ordinates be given.

Here $x=5$, and the area $= A+\frac{5}{2}B+\frac{25}{3}C+\frac{125}{4}D+125E+\frac{3125}{6}F$; and proceeding as before, we ſhall find the area $= \frac{19\times\overline{a+f}+75\times\overline{b+e}+}{288}$

$$\frac{50\times\overline{c+d}}{288}\,L.$$

For $a = A$

$$\begin{aligned}
b &= A+B+C+D+E+F\\
c &= A+2B+4C+8D+16E+32F\\
d &= A+3B+9C+27D+81E+243F\\
e &= A+4B+16C+64D+256E+1024F\\
f &= A+5B+25C+125D+625E+3125F.
\end{aligned}$$

Then for $\frac{19}{288}a$, $\frac{75}{288}b$, $\frac{50}{288}c$, $\frac{50}{288}d$, $\frac{75}{288}e$, and $\frac{19}{288}f$; put their reſpective values, from the equations above, and we ſhall have

$$\begin{array}{l}
\frac{19}{288}A\\
+\frac{75}{288}A+\frac{75}{288}B+\frac{75}{288}C+\frac{75}{288}D+\frac{75}{288}E+\frac{75}{288}F.\\
+\frac{50}{288}A+\frac{100}{288}B+\frac{200}{288}C+\frac{400}{288}D+\frac{800}{288}E+\frac{1600}{288}F\\
+\frac{50}{288}A+\frac{150}{288}B+\frac{450}{288}C+\frac{1350}{288}D+\frac{4050}{288}E+\frac{12150}{288}F.\\
+\frac{75}{288}A+\frac{300}{288}B+\frac{1200}{288}C+\frac{4800}{288}D+\frac{19200}{288}E+\frac{76800}{288}F.\\
+\frac{19}{288}A+\frac{95}{288}B+\frac{475}{288}C+\frac{2375}{288}D+\frac{11875}{288}E+\frac{59375}{288}F.
\end{array}$$

The ſum of all theſe is the area, viz.

$$\frac{288}{288}A+\frac{720}{288}B+\frac{2400}{288}C+\frac{9000}{288}D+\frac{36000}{288}E+\frac{150000}{288}F,$$

$$\text{or } A + 2\tfrac{1}{2}B + 8\tfrac{1}{3}C + 31\tfrac{1}{4}D + 125E + 520\tfrac{5}{6}F,$$

$=$ the area, as it ought to be. And thus you proceed to find the area, when more ordinates are given.

Cor. 1. *If we put* A *for the ſum of the firſt and laſt ordinates,* B *the ſum of the ſecond and laſt but one,* C *the ſum of the third and laſt but two, &c. and where the number of the ordinates is odd, let the laſt capital denote the middle ordinate. Alſo put* L *for the length of the baſe, or the diſtance of the firſt and laſt ordinates; you'll have the following table of areas againſt the reſpective number of ordinates,*

Ordinates	*Areas*
1	AL
2	$\frac{A}{2}\times L$
3	$\frac{A+4B}{6}\times L$
4	$\frac{A+3B}{8}\times L$
5	$\frac{7A+32B+12C}{90}\times L$
6	$\frac{19A+75B+50C}{288}\times L$
7	$\frac{41A+216B+27C+272D}{840}\times L$
8	$\frac{751A+3577B+1323C+2989D}{17280}\times L$
9	$\frac{989A+5888B-928C+10496D-4540E}{28350}\times L$

Cor. 2. *After the ſame manner, if a, b, c, d, &c. repreſent any equidiſtant ſections of a ſolid*; A *the ſum of the firſt and laſt*, B *the ſum of the ſecond and laſt but one, &c. the foregoing table will give the ſolidity, againſt the proper number of ſections, very nearly.*

Cor. 3. *To reduce the curve to another with fewer ordinates; take the ſum of the firſt and laſt for a new ordinate, the ſum of the ſecond and laſt but one, for another new ordinate next it, &c. and laſtly, double the middle ordinate if odd, for the laſt of them. You will have a new curve with fewer ordinates; whoſe area is eaſier found, and is nearly equal to the area of the former.*

Moreover, when the ordinates of this new curve are had; if there be taken the ſum of the firſt and ſecond ordinates, the ſum of the third and fourth, the ſum of the fifth and ſixth, and ſo on. Or *if you take the ſum of the three firſt, and the ſum of the three next, and ſo on; or if there be taken the ſum of each four, or the ſum of each five ordinates, &c. for the ordinates of another curve, it's area will be equal to the area of the given one. And in each curve the baſe muſt be diminiſhed in proportion to the number of ordinates that were added together: and thus a proper number of ordinates, in any curve being given; its quadrature will be reduced to the quadrature of another curve, with fewer ordinates.*

Cor. 4. Or *when nine ordinates give not the area exact enough, erect more; and divide the area into ſeveral parts, and find all the parts ſeparately. And it will be moſt expeditiouſly done, if you take no more than three or four ordinates at once.*

PROP. X.

To reſolve Problems by the differential Method.

RULE.

This is done by making uſe of the foregoing Propoſitions, according as the nature of the queſtion requires.

When the firſt of any order of differences is wanted, it is to be found by Prop. I.

When any term of a ſeries is required, it will be had by Prop. II.

If you want the ſum of any number of terms, it is found by Prop. III.

To interpolate any term of a ſeries, then Prop. V, VI, VII, VIII, and their Corollaries, muſt be made uſe of, as occaſion requires.

And to find the areas of curves; proceed by Prop. IX.

When the ſeveral orders of differences happen to be very great, it will be convenient to take the Logarithms of the quantities concerned, whoſe differences will be ſmaller; and at laſt when the Logarithm is obtained, the number or quantity belonging to it, and which was required, will eaſily be had.

In working by the differences of quantities, the labour may in many caſes be abridged, by putting a number of cyphers, for ſo many terms at the beginning of the ſeries; by which means you'll get ſeveral of the differences equal to 0, and ſo the anſwer will be obtained in fewer terms.

Ex. 1.

To find the firſt of the 8th order of differences in the geometrical progreſſion 1, 3, 9, 27, 81, *&c.*

Here (by Prop. I.) $n = 8$, A, b, c, &c. $= 1, 3, 9$, &c. whence $T = 1 - 8b + 28c - 56d$, &c. $= 1 - 8 \times 3 + 28 \times 9 - 56 \times 27 + 70 \times 81 - 56 \times 243 + 28 \times 729 - 8 \times 2187 + 6561 = 32896 - 32640 = 256$; that is, 256 is the firſt of the eight order of differences.

Ex. 2.

What is the firſt of the 5th order of differences, of this ſeries, $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, *&c.*

Put A, b, c, d, &c. $= 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, &c. $n = 5$. then (Pr. I.) $-T = 1 - 5b + 10c - 10d + 5e - f = 1 - 2\frac{1}{2} + 2\frac{1}{2} - 1\frac{1}{4} + \frac{5}{16} - \frac{1}{32} = 3\frac{13}{16} - 3\frac{25}{36} = \frac{1}{36}$; that is, $T = -\frac{1}{36}$.

Ex. 3.

What is the firſt of the fourth order of differences of the ſeries of cubes, 1, 8, 27, 64, 125, *&c.*

Let A, b, c, d, &c. $= 1, 8, 27, 64$, &c. $n = 4$. Then (Prop. I.) $T = A - 4b + 6c - 4d + e = 1 - 32 + 162 - 256 + 125 = 288 - 288 = 0$, ſo that $T = 0$, the fourth order of differences.

Ex. 4.

What is the 20th term of the ſeries 1, 8, 27, 64, 125, *&c.*

Here put 1, 7, 12, 6, for A, B, C, D; $n = 20$; then (Prop. II.) the 20th term will be $= A + 19B + \frac{19.18}{2}C + \frac{19.18.17}{2.3}D = 1 + 133 + 2052 + 5814 = 8000$, the 20th term of the ſeries.

A	B	C	D	
1	7	12	6	0
8	19	18	6	
27	37	24		
64	61			
125				

Ex. 5.

What is the 20th term of the triangular numbers, 1, 3, 6, 10, 15, 21, *&c.*

Here A, B, C, D, are equal to 1, 2, 1, 0; and $n = 20$; therefore (by Prop. II.) the 20th term $= A + 19B + 19 \times 9C = 1 + 38 + 171 = 210$.

A	B	C	D
1			
	2		
3		1	
6	3	1	0
10	4	1	0
15	5		

Ex. 6.

To find the ſum of n terms of the arithmetical progreſſion 1, 2, 3, 4, 5, *&c.*

Put a, d', $d'' = 1, 1, 0$, then (by Prop. III.) the ſum of n terms of the ſeries is $n + n.\frac{n-1}{2} = n + \frac{1}{2}nn - \frac{1}{2}n = \frac{nn+n}{2} = \frac{n+1}{2}n$.

A	d'	d''
1		
	1	
2		0
	1	
3		0
	1	
4		

Ex. 7.

To find the ſum of n terms of the ſquare numbers, 1, 4, 9, 16, 25, 36, *&c.*

Here a, d', d'', $d''' = 1, 3, 2, 0$; whence (by Prop. III.) the ſum of n terms is $= n + 3n \times \frac{n-1}{2} + \frac{n \times \overline{n-1} \times \overline{n-2}}{3} = \frac{2n+3nn-3n}{2} + \frac{nn-3n+2}{3}n = \frac{9n-3}{6}n + \frac{2nn-6n+4}{6}n =$

a	d'	d''	d'''
1			
4	3	2	
9	5	2	0
16	7	2	0
25	9		

$\frac{2nn+3n+1}{6}n = n\times\frac{n+1}{2}\times\frac{2n+1}{3}$, the ſum of n terms.

Ex. 8.

To find the ſum of n terms of the cube numbers, 1, 8, 27, 64, 125, *&c.*

Here ſubſtituting the values of a, d', d'', d''', by Prop. III. we have the ſum of n terms $= n+n\times\frac{n-1}{2}\times 7+12n\times\frac{n-1}{2}\times\frac{n-2}{3}+6n\times\frac{n-1}{2}\times\frac{n-2}{3}\times\frac{n-3}{4}$ $= n+\frac{7nn-7n}{2}+2n^3-6n^2+4n+\frac{n^4-6n^3+11n^2-6n}{4}$ $= \frac{n^4+2n^3+n^2}{4}$, the ſum of n terms.

a	d'	d''	d'''	
1				
	7			
8		12		
	19		6	
27		18		0
	37		6	
64		24		
	61			
125				

Ex. 9.

To find the ſum of n terms of the biquadrate numbers, 1, 16, 81, 256, *&c.*

Here ſubſtituting for a, d', d'', d''', &c. their values 0, 1, 14, 36, &c. and the ſum of n terms of the ſeries 1, 16, 81, &c. being the ſame as the ſum of $n+1$ terms of the ſeries 0, 1, 16, &c. therefore (by Prop. III.) ſubſtituting $n+1$ for n, the

a	d'	d''	d'''	d''''	
0					
	1				
1		14			
	15		36		
16		50		24	
	65		60		0
81		110		24	
	175		84		
256		194			
	369				
625					

the ſum will be $\frac{nn+n}{2}+\frac{n^3-n}{6}\times 14+\frac{n^4-2n^3-n^2+2n}{24}\times 36+\frac{n^5-5n^4+5n^3+5n^2-6n}{120}\times 24=\frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n$, the ſum of n terms required, of the ſeries 1, 16, 81, &c.

Ex. 10.

To find the ſum of n *terms of the ſeries* 1, 6, 20 50, 105, 196, 336, 540, 825, *&c.*

This is the ſame thing as finding the ſum of $n+3$ terms of the ſeries 0, 0, 0, 1, 6, 20, 50, &c. Therefore writing 0 for a, d', d''; and 1, 2, for d''', d'''', and $n+3$ for n; we have (by Prop. III.) the ſum

a	d'	d''	d'''	d''''
0	0	0	1	2
0	0	1	3	2
0	1	4	5	
1	5	9		
6	14			
20				

$$=\frac{\overline{n+3}\times\overline{n+2}\times\overline{n+1}\times n}{2.3.4}+\frac{\overline{n+3}\times\overline{n+2}\times\overline{n+1}\times n\times\overline{n-1}}{2.3.4.5}\times 2=$$

$\frac{2n^5+15n^4+40n^3+45n^2+18n}{120}$ = the ſum of n terms of the ſeries 1, 6, 20, 50, &c.

Ex. 11.

Given the logarithms of 101, 102, 104, *and* 105; *to find the log. of* 103.

Here are 4 quantities given; therefore by Prop. VI. againſt 4 you have $a-4b+6c-4d+e=0$; where c is ſought, and all the reſt are given; therefore we have $c=\frac{4\times\overline{b+d}-\overline{a+e}}{6}$.

$$\begin{array}{rr}
a = & 2.0043214 \\
b = & 2.0086002 \\
d = & 2.0170333 \\
e = & 2.0211893 \\
\hline
b+d = & 4.0256335 \\
 & 4 \\
\hline
4\times\overline{b+d} = & 16.1025340 \\
a+e = & 4.0255107 \\
\hline
6) & 12.0770233 \\
 & 2.0128372 \text{ the log. } 103.
\end{array}$$

Ex. 12.

There are given the cube roots of 45, 46, 47, 48, *and* 49; *to find the cube root of* 50.

Here five quantities are given; whence (by Prop. VI.); againſt 5 you have $a-5b+10c-10d+5e-f=0$, where f is required; therefore $f=a-5b+10c-10d+5e$.

N.	Log.
45	3.556893 = a.
46	3.583048 = b.
47	3.608826 = c.
48	3.634241 = d.
49	3.659306 = e.

$$\begin{array}{rr}
a = & 3.556893 \\
10c = & 36.088260 \\
5e = & 18.296530 \\
\hline
 & 57.941683 \\
\hline
 & 17.915240 \\
 & 36.342410 \\
\hline
 & 54.257650 \\
\hline
f = & 3.684033 = \text{cube root of } 50.
\end{array}$$

Ex. 13.

Given the natural tangents of 88°, 54′, 55′, 56′, 57′, 58′, 59′, *and* 89°; *to find that of* 88°, 58′, 18″.

Tangents	B	C	D	E	F	G
$A = 52.080673$	801436	25042	1193	76	7	1
$b = 52.882109$	826478	26235	1269	83	8	
$c = 53.708587$	852713	27504	1352	91		
$d = 54.561300$	880217	28856	1443			
$e = 55.441517$	909073	30299				
$f = 56.350590$	939372					
$g = 57.289962$						

Here taking 88° 54′ from 88° 58′ 18″, the remainder is 4′ 18″ $= 4.3$; therefore put $x = 4.3$, and (by Prop. V.) $y = A + 4.3 \times B + \frac{4.3 \times 3.3}{2} C +$

$$\frac{4.3 \times 3.3 \times 2.3}{6} D + \frac{4.3 \times 3.3 \times 2.3 \times 1.3}{24} E +$$

$$\frac{4.3 \times 3.3 \times 2.3 \times 1.3 \times .3}{120} F +$$

$$\frac{4.3 \times 3.3 \times 2.3 \times 1.3 \times .3 \times -.7}{720} G.$$

and collecting the terms,

$$\begin{array}{r} 52.080673 \\ 3.446175 \\ 177673 \\ 6489 \\ 134 \\ \hline \end{array}$$

$$\text{Tan. } 88\ 58'\ 18'' = \underline{55.711144}$$

Ex. 14.

Given the log. ſines of 1° 0′, 1° 1′, 1° 2′, 1° 3′; *to find the ſine of* 1° 1′ 40″.

Sines	B	C	D
$A = 8.2418553$	71779	$-$ 1168	38.
$b = 8.2490332$	70611	$-$ 1130	
$c = 8.2560943$	69481		
$d = 8.2630424$			

Then ſubtracting 1° 0′ from 1° 1′ 40″, there remains 1′ 40″ $= 1\frac{2}{3} = x$; then (Prop. V.) the ſine to be interpolated is $y = A + \frac{5}{3}B + \frac{5}{9}C - \frac{5}{81}D$; that is,

$$\begin{array}{rr} + & 8.2418553 \\ + & 119631 \\ - & 649 \\ - & 2 \\ \hline + & 8.2538184 \\ - & 651 \\ \hline \end{array}$$

$$\text{S. } 1^\circ\ 1'\ 40'' = \underline{8.2537533}$$

Ex. 15.

Given the ſeries $\frac{1}{50}, \frac{1}{51}, \frac{1}{52}, \frac{1}{53}, \frac{1}{54}$; *to find that which ſtands in the middle between* $\frac{1}{52}$ *and* $\frac{1}{53}$.

To ſave the trouble of reduction, take the logarithms of the numbers.

	Log. A	B	C	D	E
$\frac{1}{50}$.	$-$ 1.6989700	$-$ 86002	1671	$-$ 66	8.
$\frac{1}{51}$.	$-$ 1.7075702	$-$ 84331	1605	$-$ 58	
$\frac{1}{52}$.	$-$ 1.7160033	$-$ 82726	1547		
$\frac{1}{53}$.	$-$ 1.7242759	$-$ 81179			
$\frac{1}{54}$.	$-$ 1.7323938				

Then put $x = 2\frac{1}{2}$ the diſtance of the number to be interpolated, from $\frac{1}{50}$; then by Prop. V. $y = A + \frac{5}{2}B + \frac{15}{8}C + \frac{5}{16}D - \frac{5}{128}E$. In numbers

$$\begin{array}{rr} - & 1.6989700 \\ - & 215005 \\ + & 3133 \\ - & 21 \\ - & 1 \\ \hline - & 1.7204727 \\ + & 3133 \\ \hline - & 1.7201594 \end{array}$$

being the log. of $\frac{1}{52.5}$ the number ſought.

Fig.

Ex. 16.

Given three diſtances of the ſun from the tropic, and the times of obſervation. To find the time of the ſolſtice. 8.

viz. Dec. 20 . . . 39″
21 . . . 6
24 . . . 74.

Comparing this with Prop. VII. we have $a = 39$, $b = 6$, $c = 74$; and $p = 1$, $q = 3$; then $B = \frac{b-a}{p} = -33$, $B1 = \frac{c-b}{q} = \frac{68}{3} = 22\frac{2}{3}$; $C = \frac{B1-B}{p+q} = \frac{55\frac{2}{3}}{4} = 13\frac{11}{12}$; then (Cor. Pr. VII.) $y = 39-33x+13\frac{11}{12}x\times\overline{x-1}$, or $y = 39-46\frac{11}{12}x+13\frac{11}{12}xx = 0$, becauſe y is nothing at the time of the ſolſtice. Therefore $xx-3.3713x = -2.8024$, and $x-1.6856 = \pm\sqrt{.1383} = .3724$, and $x = 1.3132$, or $x = 2.0580$.

Ex. 17.

Given a comet's diſtance from the ſun on the following days at 12 *at night, to find its diſtance Dec.* 20.

Decem. 12 . . . diſt. 301 $= a$
21 . . . . . . 620 $= b$
24 . . . . . . 715 $= c$
26 . . . . . . 772 $= d$

Here $x = 8$, the time after Dec. 12; and by Pr. VII. $p = 9$, $q = 3$, $r = 2$; and $B = \frac{319}{9} = 35.4444$; $B1 = \frac{95}{3} = 31.6666$, $B2 = \frac{57}{2} = 28.5$; $C = -\frac{3.7778}{12} = -.3148$, $C1 = -\frac{3.1666}{5} = -.6333$ $D = -\frac{.3185}{14} = -.0227$; then

$y = 301 + 35.4444 \times 8 - 3148 \times 8 \times -1 - .0227 \times -1 \times -4 = 301 + 283.5555 + 2.5184 - .0908$; that is, $y = 586.9831$ the comet's diſtance Dec. 20, at 12 at night.

Ex. 18.

Given five places of a comet; to find it Nov. 16*d.*—17*h.*

	d.	h.				
Nov.	3	17,	place	Leo	29	51
	5	15	. .	Virgo	3	23
	10	16	. . . . .		15	32
	18	21	. .	Libra	18	52
	20	17	. . . . .		28	10

Proceeding by Prop. VII. we ſhall have

	d.	h.				
Nov.	3	17				
			1	22	$= 1.9166$	$= p.$
	5	15				
			5	1	$= 5.0417$	$= q.$
	10	16				
			8	5	$= 8.2083$	$= r.$
	18	21				
			1	20	$= 1.8333$	$= s.$
	20	17				

Leo	29	51	$= a$				
				3	32	$= 3.5333$	$= b-a.$
Virgo	3	23	$= b$				
				12	9	$= 12.15$	$= c-b.$
	15	32	$= c$				
				33	20	$= 33.3333$	$= d-c.$
Libra	18	52	$= d$				
				9	18	$= 9.3$	$= e-d.$
	28	10	$= e$				

$B = 1.8436$, $C = .0814$, $D = .0028$, $E = -.0003$
$B1 = 2.4099$, $C1 = .1246$, $D1 = -.0016$
$B2 = 4.0609$, $C2 = .1008$
$B3 = 5.0729$

from Nov. 16 17
take Nov. 3 17

remains 13 0 therefore $x = 13$, whence

(Cor. 1. Pr. VII.) $y = a + 1.8436 \times 13 + .0814 \times 13 \times 11.0834 + .0028 \times 13 \times 11.0834 \times 6.0417 - .0003 \times 13 \times$

$\times 11.0834 \times 6.0417 \times -2.1666 = a + 38^\circ.6982 =$ Leo $29^\circ\ 51' + 38^\circ\ 41'\ 53'' =$ Libra $8^\circ\ 32'\ 53''$, its place November 16d. 17h.

Ex. 19.

In the given ſeries $1, \frac{2}{1}\times 1, \frac{4}{3}\times\frac{2}{1}\times 1, \frac{6}{5}\times\frac{4}{3}\times\frac{2}{1}\times 1, \frac{8}{7}\times\frac{6}{5}\times\frac{4}{3}\times\frac{2}{1}\times 1$, *&c. to find the term in the middle between the ſecond and third.*

$$\begin{array}{cccccc}
A & B & C & D & E & \&c. \\
1 & & & & & \\
 & 1 & & & & \\
2 & & -\frac{1}{3} & & & \\
 & \frac{2}{3} & & \frac{1}{5} & & \\
2\frac{2}{3} & & -\frac{2}{15} & & -\frac{1}{7} & \\
 & \frac{8}{15} & & \frac{2}{35} & & \\
3\frac{1}{5} & & -\frac{8}{105} & & & \\
 & \frac{16}{35} & & & & \\
3\frac{23}{35} & & & & &
\end{array}$$

By Prop. V. $x = 1\frac{1}{2}$. Alſo $P = \frac{3}{2}$; $Q = \frac{3}{8}$, $R = -\frac{1}{16}$, $S = \frac{3}{128}$, $T = -\frac{3}{256}$, $V = \frac{7}{1024}$, &c.

then $y = A + \frac{3}{2}B + \frac{3}{8}C - \frac{1}{16}D + \frac{3}{128}E - \frac{3}{256}F + \frac{7}{1024}G$ &c.

or $y = 1 + 1\frac{1}{2} - \frac{1}{8} - \frac{1}{80} - \frac{3}{896} - \frac{1}{768} - \frac{7}{11264}$ &c.

that is $y = 2.357$.

Ex. 20.

Fig. 10.

*Given the lengths of three equidiſtant ordinates (*5, 7, *and* 8*)*; *and the length of the baſe* MQ = 10; *to find the area of the curve* AMQC.

By Cor. 1. Prop. IX. A = AM + CQ = 5 + 8 = 13. B = BP = 7, then for three ordinates, the area $= \frac{A+4B}{6}L = \frac{13+28}{6}\times 10 = \frac{41}{6}\times 10 = 68\frac{1}{3}$.

Ex. 21.

11.

*Given the length of four equidiſtant ordinates (*5, 7, 8, 10*), and baſe* MR = 10, *to find the area of the curve* AMRD.

Here A = AM + DR = 15, B = BP + CQ = 15. Then for four ordinates, the area (Cor. 1. IX.) is $= \frac{A+3B}{8}L = \frac{15+45}{8}\times 10 = \frac{60}{8}\times 10 = 75$, as required.

Other-

Fig. 11.

Otherwiſe,

Put AM $= 5+10 = 15$, and BP $= 7+8 = 15$; then put A $=$ AM $+$ BP $= 30$; and the area for theſe two ordinates is $= \frac{A}{2} \times L = \frac{30}{2} \times 5 = 75$.

Ex. 22.

Given the lengths of five equidiſtant ordinates, (10, 11, 14, 16, 16), *and the length of the baſe* MS, 20; *to find the area.*

Here A $=$ AM$+$ES $= 26$, B $=$ BP$+$DR $= 27$, C $= 14$, L $= 20$. Then (by Cor. 1. Pr. IX.) the area is $\frac{7A+32B+12C}{90}L = \frac{182+864+168}{90} \times 20 = \frac{1214}{90} \times 20 = 269\frac{7}{9}$.

Or thus,

Reduce it to 3 ordinates, putting AM $= 10+16 = 26$, BP $= 11+16 = 27$, CQ $= 2\times 14 = 28$. Then A $=$ AM$+$CQ $= 54$, B $= 27$; then the area $= \frac{A+4B}{6} \times 10 = \frac{54+108}{6} \times 10 = \frac{162}{6} \times 10 = 270$.

Or thus,

In the laſt curve, the three ordinates are 26, 27, 28; make AM $= 26+27 = 53$, BP $= 27+28 = 55$, L $= 5$; and put A $=$ AM$+$BP $= 108$; then the area $= \frac{A}{2}L = \frac{108}{2} \times 5 = 270$.

Ex. 23.

12. *Given the lengths of ſix equidiſtant ordinates,* (10, 12, 11, 7, 0, 9,); *baſe* 20; *to find the area.*

Here A $= 10-9 = 1$, B $= 12+0 = 12$, C $= 11+7 = 18$, L $= 20$; then (Cor. 1. Pr. IX.), area $= \frac{19A+75B+50C}{288} \times L = \frac{19+900+900}{288} \times 20 = \frac{1819}{288} \times 20 = 126\frac{1}{24}$; where note, as the ordinate VF is negative; the area SVF below the baſe, is alſo negative.

Or thus,

Divide the area into two parts, one with four ordinates, the other with three. For the four ordinates 10, 12, 11, 7; we have A = 17, B = 23; and the area $= \frac{A+3B}{8} \times 12 = \frac{17+69}{8} \times 12 = \frac{86}{8} \times 12 = 129$.

For the three ordinates 7, 0, —9; we have A = —2, B = 0; whence the area $= \frac{A+3B}{8} L = \frac{-2}{8} \times 8 = -2$.

Then the whole area = 129—2 = 127.

Ex. 24.

Let 18 *ordinates be given in a curve,* 0, 2, 3, 4, 6, 5, 3, 1, —1, —3, —4, —5, —5, —3, —1, 0, 2, 5; *and the baſe* 17; *to find the area.*

The ſhorteſt way is to take three or four ordinates at once. Thus for the four ordinates 0, 2, 3, 4; we have A = 4, B = 2+3 = 5; and the area $= \frac{A+3B}{8} L = \frac{4+15}{8} \times 3 = \frac{57}{8} = 7\frac{1}{8}$.

Then for the area with the four ordinates 4, 6, 5, 3; here A = 4+3 = 7, B = 6+5 = 11; and the area $= \frac{7+33}{8} \times 3 = 15$.

Again, with the four ordinates 3, 1, —1, —3; here A = 0, B = 0, and the area = 0.

Then with the four ordinates -3, -4, -5, -5; here $A = -8$, $B = -9$; and the area $= \frac{-8-27}{8} \times 3 = -\frac{35}{8} \times 3 = -13\frac{1}{8}$.

Then with the four ordinates -5, -3, -1, 0; here $A = -5$, $B = -4$; and the area $= \frac{-5-12}{8} \times 3 = -\frac{17}{8} \times 3 = -6\frac{3}{8}$.

Laſtly with the three ordinates $0, 2, 5$; here $A = 5$, $B = 2$; and the area $= \frac{5+8}{6} \times 2 = \frac{13}{3} = 4\frac{1}{3}$.

Then the ſum of all theſe areas is $7\frac{1}{8} + 15 - 13\frac{1}{8} - 6\frac{3}{8} + 4\frac{1}{3} = 6\frac{23}{24}$; that is the whole area $= 6\frac{23}{24}$, eſteeming theſe parts negative, that are below the baſe.

Ex. 25.

Given 8 ordinates of a hyperbola between the aſſymptotes, $\frac{10}{10}, \frac{10}{11}, \frac{10}{12}, \frac{10}{13}, \frac{10}{14}, \frac{10}{15}, \frac{10}{16}, \frac{10}{17}$; *to find the area.*

Here $a = \frac{10}{10} = 1.00000$
$b = \frac{10}{11} = 0.90909$
$c = \frac{10}{12} = 0.83333$
$d = \frac{10}{13} = 0.76923$
$e = \frac{10}{14} = 0.71428$
$f = \frac{10}{15} = 0.66666$
$g = \frac{10}{16} = 0.625$
$h = \frac{10}{17} = 0.58823$

then $A = a+h = 1.58823$, $B = b+g = 1.53409$, $C = c+f = 1.5$, $D = d+e = 1.48351$; then (Cor. 1. Pr. IX.) the area $=$

$$\frac{751A + 3577B + 1323C29 + 89D}{17280} L, \text{ where } L = 7;$$

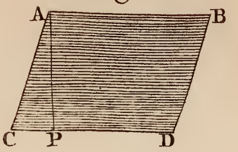
that is, the area $=$

$$\frac{1192.76 + 5487.44 + 1984.5 + 4434.21}{17280} \times 7 =$$

5.3062, as required.

THE

Fig. 1.



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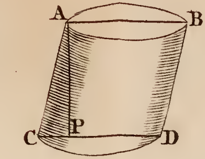

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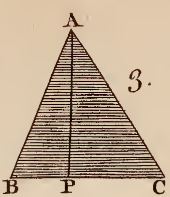

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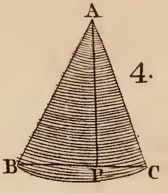

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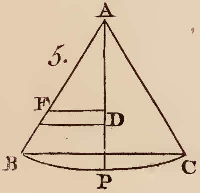

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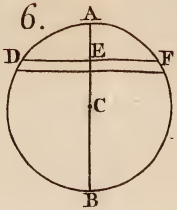

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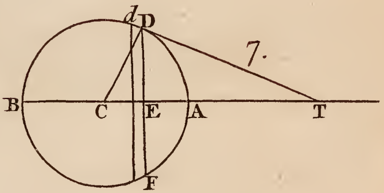

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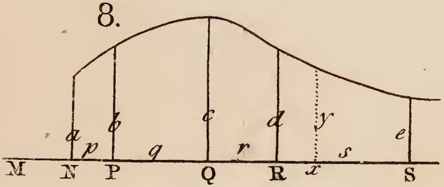

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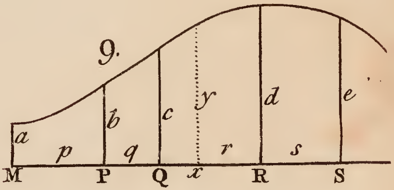

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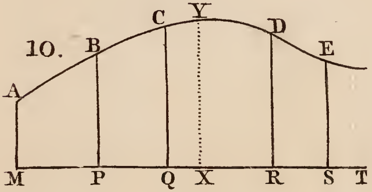

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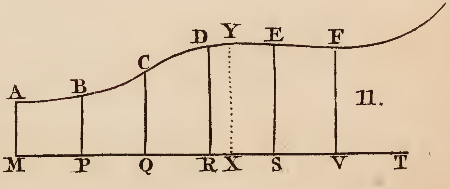

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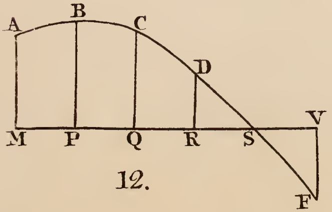


Ar. Infinites, and Dif. Method. *the End.*

2

THE
ELEMENTS
OF THE
CONIC SECTIONS;
DEMONSTRATED
In THREE BOOKS.

Book I. Of the Ellipsis.
Book II. Of the Hyperbola.
Book III. Of the Parabola.

Et ſi omnia à veteribus inventa eſſent: hoc tamen ſemper novum erit, uſus et inventorum ab aliis ſcientia et diſpoſitio.
Senec.

THE

PREFACE.

IT is not my deſign in this Treatiſe, to lay down all the properties of the Conic Sections, but ſhall confine myſelf to thoſe that are moſt uſeful. For to ſay all that has been written, or may be written concerning theſe curves, would far exceed my intended brevity, and be attended with no advantage. Therefore ſetting aſide all trifling and uſeleſs propoſitions, I have endeavoured to explain the chief and moſt conſiderable properties of theſe curves, and I think I have omited but few things that are either uſeful or curious.

The moſt uſeful and remarkable curves, next the circle, are the Conic Sections. Theſe curves make a conſiderable branch in geometry. They are of great uſe in ſeveral parts of the mathematics, particularly in Dialling, *for delineating the ſigns; likewiſe in the* Projection *of the ſphere, where ſeveral of the circles are projected into Conic Sections; and the like in* Perſpective; *alſo in* Optics, *to reflect or refract the rays to a geometrical focus; all* Aſtronomy *is built thereon; the* Conſtruction *of algebraic equations depends much upon them; they are very uſeful in natural* Philoſophy *for inveſtigating the phœnomena of nature. A man cannot be a proficient in the abſtruſer parts of the mathematics, without the knowledge of the Conic Sections, which leads the way to the higher geometry. And it is abſolutely neceſſary to know the fundamental properties, for all ſuch perſons, as would make any advances in natural philoſophy, and for underſtanding the amazing diſcoveries of the laſt age, in the mechaniſm of the univerſe.*

As to this Treatiſe, I have comprehended all I had to ſay on this ſubject, in three Books, for the three Conic Sections ſeparately: which is better, at leaſt for beginners, than treating them altogether, in a more abſtract and general manner. The ſeveral propoſitions are delivered in ſuch order, as they

they ſeemed moſt naturally to follow from one another; and, for the ſeveral Sections, are laid down nearly in the ſame order, and almoſt in the ſame words, that the relation they have to one another, may the better appear.

In the Hyperbola, *many of the propoſitions comprehend a great many particular caſes, which would require a vaſt number of cuts to deſcribe them all, as they relate to the oppoſite Sections, or to the conjugate hyperbola's. Therefore I have treated them in as general a way as I could. And any particular caſe may be eaſily inferred from the general propoſition,* mutatis mutandis, *attending duly to the nature of that propoſition. And to aſſiſt the reader's imagination, in the moſt remarkable propoſitions, I have denoted ſuch particular caſes by ſmall letters of the Alphabet. And in many inſtances I have done the ſame thing with the other Sections, where it appeared neceſſary.*

I have choſen to demonſtrate theſe their properties from their ſimple deſcription upon a plane, without making uſe of the cone, as being eaſier to underſtand. For when the demonſtrations are taken from the Sections of a cone, they are more difficult to comprehend, by reaſon there are ſo many interſections of planes with planes, and planes with ſolids, that the reader is confuſed with them. Whereas theſe properties are eaſily drawn from their mechanical deſcription upon a plane, and the demonſtrations are more ſimple and natural.

At the end of each Book I have given a collection of the moſt uſeful Problems. And laſt of all I have ſhewn what ſort of curves theſe are, which are made by cutting a cone by a plane in ſuch and ſuch poſitions; and have demonſtrated that theſe are the very ſame curves, we have here been treating on. So I hope the reader will be furniſhed here with every thing that is neceſſary for him, in this branch of geometry.

W. Emerſon.

BOOK

CONIC SECTIONS.

BOOK I.

Of the ELLIPSIS.

DEFINITIONS.

DEFIN. I.

IF two pins be fixed at the points F, S; and a thread PSFP, put about them and knotted at P; then if the thread be drawn tight, and the point P and the thread be moved about the fixed centers F, S; the point P will deſcribe the curve PDpBEAP, called an *Ellipſis*. Fig. 1.

DEF. II.

The points or centers F, S, are called the *Foci.*

DEF. III.

The line AB, drawn thro' the foci to the curve, is called the *tranſverſe Axis.*

DEF. IV.

The point C in the middle of the axis AB, is the *Center.*

DEF. V.

The line DE, (drawn thro' the center C) perpendicular to the tranſverſe AB, is called the *conjugate Axis.*

DEF. VI.

Fig. 2. Any line TO, drawn thro' the center C to the curve, is called a *Diameter*. And the extremity T (or O) its *vertex*.

DEF. VII.

If TO be a diameter, then the diameter GK, drawn parallel to the tangent at its vertex T, is called its *conjugate*. And the two diameters TO, GK, are ſaid to be *conjugates* to one another.

DEF. VIII.

The line LR (drawn thro' the focus F, perpendicular to the tranſverſe axis AB,) is called the *parameter* or *latus rectum*.

DEF. IX.

A line drawn from any point of the curve (as HI) perpendicular to the tranſverſe axis, is called an *Ordinate* to the tranſverſe. And in general, any line drawn from the curve to any diameter TO, parallel to its conjugate GK, (as HN,) is an *ordinate* to that principal diameter TO. If it go quite thro' the figure, as H*h*, it is called a *double Ordinate*.

DEF. X.

A right line meeting the ellipſis in one point M, but does not cut it, is called a *Tangent* to it in that point, as TM.

DEF. XI.

The part of the diameter between the vertex and the ordinate, is called the *Axſciſſa*: TN, AI. And the *Vertex* is the extremity of any diameter.

PROP.

PROP. I.

The ſum of the lines FP, SP, *drawn from the foci, to any point of the curve, is equal to the tranſverſe axis* AB. 1.

For by conſtruction, $PF+PS = AF+AS = AF+AF+FS = 2AF+FS$. And the ſame $PF+PS = 2BS+FS$; therefore $2AF+FS = 2BS+FS$; and $2AF = 2BS$, or $AF = BS$. Whence $PF+PS = 2AF+FS = AF+BS+FS = AB$.

Cor. *The two foci are equally diſtant from the vertexes, and alſo from the center* : $AF = BS$, and $FC = SC$.

For it is proved that $AF = BS$; and ſince $AC = CB$ (def. 4), therefore $AC-AF = CB-BS$, or $FC = SC$.

PROP. II.

A line drawn from the end of the conjugate axis, to the focus, is equal to half the tranſverſe; $DF = CA$. 3.

Draw DS to the other focus. Then the two right angled triangles CDF and CDS are ſimilar and equal. For $SC = CF$, the angles at C are right, and CD common; therefore $SD = DF$; and ſince the ſum $SD+DF =$ the tranſverſe (Prop. I.), one of them $DF =$ half the tranſverſe CA.

Cor. *The diſtance of the foci is a mean proportional between the ſum and difference of the tranſverſe and conjugate axes*, $SF^2 = \overline{BA+DE} \times \overline{BA-DE}$.

For $CA^2 = DF^2 = DC^2+CF^2$; and $CF^2 = CA^2-CD^2 = \overline{CA+CD} \times \overline{CA-CD}$; and $4CF^2$ or $SF^2 = \overline{2CA+2CD} \times \overline{2CA-2CD}$.

PROP. III.

3. *The rectangle of the focal diſtances from either vertex, is equal to the ſquare of the ſemiconjugate*: $AF\times FB = DC^2$.

For $DC^2 = DF^2 - CF^2 =$ (Prop. II.) $CA^2 - CF^2 =$ (Geom. I. 12.) $\overline{CA+CF}\times\overline{CA-CF} = \overline{BC+CF}\times\overline{CA-CF} = BF\times FA$.

PROP. IV.

3. *As the tranſverſe axis to the conjugate, ſo the conjugate to the latus rectum of the tranſverſe*: $AB : DE :: DE : LR$.

For $SL+LF = BA = 2CA$ (Prop. I.); and $SL = 2CA - LF$, and by ſquaring (Geom. I. 11.), $SL^2 = 4CA^2 - 4CA\times LF + LF^2$. And in the right angled triangle SLF, $SL^2 = SF^2 + LF^2$; whence $4AC^2 - 4CA\times LF + LF^2 = SF^2 + LF^2$, and $4AC^2 - 4CA\times LF = SF^2 = 4CF^2$, and $4AC^2 = 4CA\times LF + 4CF^2 = 4CA\times LF + 4DF^2 - 4DC^2$, and $4AC^2 + 4DC^2 = 4CA\times LF + 4DF^2$; but $CA^2 = DF^2$ (Pr. II.); therefore $4DC^2 = 4CA\times LF = 2CA\times 2LF$; that is, $DE^2 = BA\times LR$.

Cor. 1. *As the ſemitranſverſe is to the ſemi-conjugate, ſo the ſemiconjugate to half the latus rectum*; $CA : DC :: DC : LF$.

Cor. 2. *As the ſemitranſverſe, to the diſtance of the focus from the center; ſo is the ſame diſtance, to the difference between the ſemitranſverſe and half the latus rectum*: $FC^2 = CA\times\overline{CA-LF}$.

For $CF^2 = DF^2 - DC^2 =$ (Pr. II.) $CA^2 - CD^2 = CA^2 - CA\times LF$ (Pr. IV. Cor. 1.)

Cor. 3. *The rectangle* BFA $=$ *half the tranſverſe* Fig. 3.
$\times$ *half the latus rectum* $= CA\times FL$. By Cor. 1. and Prop. III.

SCHOLIUM.

Since the tranſverſe axis is to the conjugate, ſo the conjugate to the latus rectum, of the tranſverſe axis. Therefore in any other diameters, the third proportional, to the diameter and its conjugate, is called the *latus rectum* of that diameter.

PROP. V.

From any point M *in the curve, drawing the lines* MF, MS, *to the two foci; and the ordinate* MP *perpendicular to the tranſverſe axis* BA; *it will be,* 4.
As the ſemitranſverſe, CA:
to the diſtance of the focus from the center, CF::
So the diſtance of the ordinate from the center, CP:
to half the difference of the lines MF, MS, *or* $\frac{MS-MF}{2}$.

For make $SD = CA$, then $SM = CA+DM$, and $FM = 2CA-SM = CA-DM$. In the right angled triangle SMP, SM^2 or $CA^2+2CA\times DM+DM^2$ (Geom. I. 10.) $= SP^2+PM^2 = \overline{CF+CP}^2+PM^2 = CF^2+2CF\times CP+CP^2+PM^2$, and in the right angled triangle FMP, FM^2 or $CA^2-2CA\times DM+DM^2 = FP^2+PM^2 = \overline{CF-CP}^2+PM^2 =$ (Geom. I. 11.) $CF^2-2CF\times CP+CP^2+PM^2$; then ſubtracting the latter equation from the former, $SM^2-FM^2 = 4CA\times DM = 4CF\times CP$, and $CF\times CP = CA\times DM$. But ſince $SM = CA+DM$, and $FM = CA-DM$; therefore $SM-FM = 2DM$; therefore $CF\times CP = CA\times\frac{SM-FM}{2}$.

Fig. Cor. 1. *If* F, S *be the foci*, MP *an ordinate*;
4. *then it is* $CA : CF :: CP : \overline{CA-MF}$ or $\overline{SM-CA}$.

For $CF\times CP = CA\times DM$, and $DM = SM-CA = CA-FM$.

Cor. 2. *If* F, S *be the foci*, MP *an ordinate*; *then the difference of the ſquares of the lines* SM, FM; *that is*, $SM^2-FM^2 = 4CF\times CP$.

Cor. 3. *If* F, S *be the foci*, MP *an ordinate*; *then* $CA\times\overline{SM-FM} = 2CF\times CP$.

For $SM^2 - FM^2 = \overline{SM+FM}\times\overline{SM-FM}$ (Geom. I. 12.) $= 2CA\times\overline{SM-FM} = 4CF\times CP$, and $CA\times\overline{SM-FM} = 2CF\times CP$.

SCHOLIUM.

If PM falls on the other ſide of F, as *pm*, then $pF = Cp-CF$, and its ſquare the ſame as before, and the reſt of the demonſtration the ſame.

PROP. VI.

4. *If an ordinate* MP *be drawn to the tranſverſe axis*; *it will be*,
As the ſquare of the tranſverſe, BA^2 :
to the ſquare of the conjugate, NE^2 : :
So the rectangle of the ſegments of the tranſ. BPA :
to the ſquare of the ordinate, PM^2.

For make $SD = CA$, then DM is half the difference of SM and MF; therefore by Prop. V.
$CA : CF :: CP : DM$, and (Propor. 13.)
$CA : CA+CF$ or $BF :: CP : CP+DM$, and (Pro. 4.)
$CA : CP :: BF : CP+DM$, and (Propor. 13.)
$CA : CA+CP$ or $BP :: BF : BF+CP+DM$.

But $BF = BC+CF = SD+CF$; and $BF+CP+DM = SD+CF+CP+DM = SM+CS+CP = SM+SP$; whence
$CA : BP :: BF : SM+SP$.

Again,

Fig. 4.

Again, ſince $CA : CF :: CP : DM$; then (Pro. 13.) $CA : (CA - CF)\,AF :: CP : CP - DM$; and (Propor. 4.) $CA : CP :: AF : CP - DM$, and (P. 13.) $CA : (CA - CP)\,PA :: AF : AF - CP + DM$.

But $AF = CA - CF = SD - SC$; therefore $AF - CP + DM = SD - SC - CP + DM = SM - SP$; therefore

$CA : PA :: AF : SM - SP$, and we had before, $CA : BP :: BF : SM + SP$; then multiplying theſe proportions together (Propor. 18), we have $CA^2 : BP \times PA :: BF \times FA : SM^2 - SP^2$ (Geom. 12.)

But (Prop. III.) $BF \times FA = CN^2$, and Geom. II. 21. Cor. 1.) $SM^2 - SP^2 = PM^2$; therefore $CA^2 : BPA :: CN^2 : PM^2$, or alternately

$$CA^2 : CN^2 :: BPA : PM^2.$$

or $BA^2\ (4CA^2) : NE^2\ (4CN^2) :: BPA : PM^2$.

Cor. 1. $CA^2 : CN^2 :: BFA : PM^2$.

Cor. 2. *As the tranſverſe* BA :
to its latus rectum : :
So the rectangle BPA :
to ſquare of the ordinate PM^2.

For (Prop. IV.) latus rectum $= \frac{NE^2}{AB}$, whence ſince

$$BA^2 : EN^2 :: BFA : PM^2.$$

therefore $BA : \frac{NE^2}{BA}$ or latus rectum $:: BFA : PM^2$ (Propor. 6.).

Cor. 3. *The rectangles of the ſegments of the tranſverſe, are as the ſquares of the ordinates.*

For every rectangle is to the ſquare of its ordinate, in the given ratio of CA^2 to CN^2, or of BA to the latus rectum.

Cor. 4. *As the ſquare of the ſemitranſverſe* CA^2 :
Rectangle of the focal diſtances from vertex BFA : :
So rectangle of the ſegments BPA :
Square of the ordinate PM^2.

Fig.

PROP. VII.

5. *If the ordinate* MG *be drawn perpendicular to the conjugate* DE, *it will be,*

As the ſquare of the ſemiconjugate, DC^2 :
to ſquare of the ſemitranſverſe, CA^2 : :
So the rectangle of the ſegments of conj. DGE :
to ſquare of the ordinate, GM^2.

For by Cor. 1. Pr. VI. $CA^2 : CD^2 :: BPA : PM^2$. But $BPA = \overline{AC+CP} \times \overline{AC-CP}$ (Geom. I. 12.) $= AC^2 - CP^2 = AC^2 - GM^2$; therefore $CA^2 : CD^2 :: AC^2 - CP^2 : PM^2$, and alternately $CA^2 : CA^2 - CP^2 :: CD^2 : PM^2$; and by diviſion $CA^2 : CP^2 :: CD^2 : CD^2 - PM^2$ or $CD^2 - CG^2$; and alternately $CA^2 : CD^2 :: CP^2 :: CD^2 - CG^2$ or $\overline{CD+CG} \times \overline{CD-CG}$ that is, $EG \times GD$; or $CA^2 : CD^2 :: GM^2 : DGE$.

Cor. *The whole ordinate to the tranſverſe or conjugate axis, is biſſected by that axis; and equal ordinates are equally diſtant from the center.*

For the rectangle of the ſegments of the axis being in a given ratio to the ſquare of the ordinate; and that rectangle being the ſame, for the ordinates on both ſides; theſe ordinates muſt be equal. Likewiſe that rectangle remains the ſame, at equal diſtances on each ſide the center; and therefore the ordinates are equal at equal diſtances.

PROP. VIII.

6. *If* MP *be an ordinate, to the tranſverſe*; F, S *the foci*; *and you make* $CA : CF :: CP : CI$; *then the rectangle* $SM \times MF = CA^2 - CI^2$.

For make $SD = CA$, then $DM = \dfrac{SM - FM}{2}$, and by Prop. V. $DM = \dfrac{CF \times CP}{CA} = CI$ by conſtruction;

ſtruction; and $SM\times MF=\overline{CA+DM}\times\overline{CA-DM}=CA^2-DM^2=CA^2-CI^2$. Fig. 6.

Cor. 1. *If* $CA:CF::CP:CI$; *then* $FM=AI$, *and* $SM=BI$.

For $DM=CI$, and $SM=CA+DM=CA+CI=BI$, and therefore $FM=AI$.

Cor. 2. $CA:CF::CP:CA-FM$ *or* $SM-CA$.

PROP. IX.

If F, S *be the foci*, C *the center*, MP *an ordinate to the tranſverſe*, CE *the ſemiconjugate*; *then* $CM^2=CA^2+CE^2-SMF$. 6.

For in the triangle SMF, (Geom II. 28), $SM^2+MF^2=2CF^2+2CM$. But $\overline{SM+MF}^2=SM^2+2SM\times MF+MF^2$; therefore $SM^2+MF^2=\overline{SM+MF}^2-2SM\times MF=BA^2-2SM\times MF$; whence $BA^2-2SM\times MF=2CF^2+2CM^2$. Therefore $2CM^2=BA^2-2CF-2SM\times MF=4CA^2-2CF^2-2SM\times MF$; and $CM^2=2CA^2-CF^2-SM\times MF=CA^2+CE^2-SM\times MF$.

Cor. 1. *If* $CA:CF::CP:CI$; *then* $CM=EI$. It follows from this and the laſt Prop.

Cor. 2. *If* $CA:CF::CP:CI$; *then rectangle* BPA $=$ *rectangle* $SIF+PM^2$.

For $BPA=CA^2-CP^2=CA^2-CM^2+PM^2=SMF-CE^2+PM^2=$ (VIII) $CA^2-CI^2-CE^2+PM^2=CF^2-CI^2+PM^2=SIF+PM^2$.

PROP. X.

If from the foci, two lines SM, FM *be drawn to any point in the curve, as* M; *they will make equal angles with the curve in that point.* 7.

Take *m* infinitely near M in the curve, and draw S*m*, F*m*; and from S and F deſcribe the ſmall arches

Fig. 7. arches Mr, mt, thro' M and m. Then ſince $SM+MF = Sm+mF$, by Pr. I. that is, $SM+Mt+tF = Sr+rm+mF$; and taking away the equal quantities SM, Sr, as alſo tF, mF; there remains $Mt = rm$. Therefore in the infinitely ſmall triangles Mrm, and Mtm, right angled at r and t, we have $Mt = mr$, and the hypothenuſe Mm common; therefore the ſides Mr and mt are equal, and the oppoſite angle Mmr and mMt; that is, MmS and mMF are equal; but m being infinitely near M, the angle $MmS = GMS$ the external angle at M; whence $mMF = GMS$.

Cor. *If a line* MO *be drawn perpendicular to the tangent* MT *in* M; *the angles* FMO *and* SMO *are equal.*

For they are the complements of the equal angles FMT and SMG.

PROP. XI.

8. *If* MP *be an ordinate to the tranſverſe; and* MO *perpendicular to the tangent at* M; *and* S, F *the foci; then* $CA^2 : CF^2 :: CP : CO$.

For (Geom. II. 25.) in the triangle SMF, where MO biſſects the angle SMF, it is $SM : MF :: SO : OF$; and by compoſition (Propor. 13.) $SM+MF : SM-MF :: SO+OF : SO-OF$; that is, $2CA : SM-MF :: 2CF : 2CO :: CF : CO$, and $CA^2 : \frac{SM-MF}{2} \times CA :: CF : CO$. But (Prop. V.) $\frac{SM-MF}{2} \times CA = CF \times CP$; therefore $CA^2 : CF \times CP :: CF : CO :: CF^2 : CF \times CO$. And alternately $CA^2 : CF^2 :: CF \times CP : CF \times CO :: CP : CO$.

Cor. 1. $CA^2 : CD^2 :: CP : PO$.

For $CA^2 : CF^2 :: CP : CO$; and by diviſion, Fig.
$CA^2 : CA^2 - CF^2$ or $CD^2 :: CP : CP - CO$ or PO. 8.

Cor. 2. $2CA$: *latus rectum* :: CP : PO.

For $CA^2 : CD^2 :: CP : PO$, and $2CA : \frac{2CD^2}{CA}$, or the latus rectum : CP : PO.

Cor. 3. CA : MF :: CF : OF.

For we had $2CA : SM - MF :: CF : CO$, and by diviſion $2CA : 2CA - SM + MF :: CF : CF - CO$; that is, $2CA : SM + MF - SM + MF :: CF : OF$; that is, $2CA : 2MF$, or $CA : MF :: CF : FO$.

PROP. XII.

If S, F *be the foci, and* MO *perpendicular to the tangent in* M; *then* $MO^2 = SM \times MF - SO \times OF$. 8.

For ſince MO biſſects the angle SMF; therefore (Geom. II. 26.) $MO^2 + SOF = SMF$; and $MO^2 = SMF - SOF$.

Cor. 1. $MO = \frac{CD}{CA}\sqrt{SM \times MF}$; or $CA^2 : CD^2 :: SMF : MO^2$.

For (Geom. II. 25.) $SM : MF :: SO : OF$, and by compoſition, $SM + MF$ or $2CA : MF :: SO + OF$ or $SF : OF$.

or $CA : CF :: MF : OF$, in like
manner $CA : CF :: SM : SO$, and multiplying,
$CA^2 : CF^2 :: SMF : SOF$;
and $CA_2 : CA^2 - CF^2$ or $CD^2 :: SMF : SMF - SOF$.
that is, $CA^2 : CD^2 :: SMF : MO^2$, by this Prop.

Cor. 2. $SMF = MO^2 + BOA - CD^2$.

For $SMF = MO^2 + SOF$. But $SOF = SC^2 - CO^2 = BC^2 - CD^2 - CO^2 = BOA - CD^2$.

PROP. XIII.

9. *If thro' any point* M, *the line* SG, *equal to* BA, *be drawn from the focus* S, *and* FG *from the other focus* F; *and the line* MD *be drawn to the middle of* FG; *then* MD *will touch the curve in* M.

For draw FM, then $MG = AB - SM = FM$; therefore MD is perpendicular to FG, whence $FM = GM$.

To any other point *m*, in the line DM; draw S*m*, G*m*, F*m*; then in the triangles G*m*D, F*m*D; the angles at D being right, and the including ſides equal, the hyp. $mG = mF$. In the triangle SG*m*, the ſum of the two ſides $Sm + mG$ is greater than the baſe $SM + MG$, (Geom. II. 5.); that is, $Sm + mF$ is greater than $SM + MF$; and therefore by conſtruction of the ellipſis, the point *m* is without it; and ſo are all points except the point M.

Cor. 1. *The tangent at* M, *biſſects the external angle* FMG, *made by two lines drawn from the focus.*

Cor. 2. *The tangents at* A *and* B *are perpendicular to the tranſverſe axis* AB; *likewiſe a tangent at the end of the conjugate is perp. to the conjugate.*

PROP. XIV.

10. *If* PM *be an ordinate,* AD *a line drawn from the ends of the tranſverſe and conjugate axes, and* PQ *parallel to* AD; *then* $PM^2 = CD^2 - CQ^2 = DQ \times QE$.

For by the ſimilar triangles ACD, PCQ; $AC : CD :: PC : CQ$; and $AC^2 : CD^2 :: PC^2 : CQ^2$; and (Propor. 13. laſt caſe) $AC^2 : CD^2 :: AC^2 - PC^2 : CD^2 - CQ^2$. But (Prop. VI.), $AC^2 : CD^2 :: BPA$

or $AC^2 - PC^2 : PM^2$; therefore $PM^2 = CD^2 - CQ^2$ Fig.
or $\overline{CD+CQ} \times \overline{CD-CQ}$.

PROP. XV.

If MP *be an ordinate to the tranſverſe axis, and* MH *be drawn perpendicular to the curve at* M, *cutting the tranſverſe and conjugate axes at* O *and* H. *It will be as* $OP : OC :: CD^2 : CF^2$. F *being the focus.* 11.

For (by Prop. XI.) $CA^2 : CF^2 :: CP : CO$. And by diviſion, $CA^2 - CF^2 : CF^2 :: CP - CO : CO$; that is, $CD^2 : CF^2 :: OP : CO$.

Cor. 1. $CD^2 : CF^2 :: MO : OH :: MP : CH :: PO : CO$.

This follows from the ſimilar triangles MOP and HOC.

Cor. 2. *If* L *be the latus rectum of* AB; *then,* $L : AB - L :: MO : OH :: OP : OC :: MP : CH$.

For $CD^2 : CF^2$ or $CA^2 - CD^2 :: MO : OH$, &c. (by Cor. 1.); therefore $\frac{CD^2}{CA} : CA - \frac{CD^2}{CA} :: MO : OH$ (Propor. 5. Cor. 1.); that is, (by Pr. IV. Cor. 1.) $\frac{1}{2}L : CA - \frac{1}{2}L :: MO : OH$; or $L : BA - L :: MO : OH$, &c.

Cor. 3. $CA^2 : CD^2 :: CH + PM : PM :: CP : OP$.

For $CD^2 : CF^2 :: MP : CH$; and compounding, $CD^2 : CD^2 + CF^2$ (CA^2) $:: MP : MP + CH :: CP : OP$.

PROP. XVI.

If PM *be an ordinate to the tranſverſe,* MT *a tangent at* M; *then* CP, CA, CT *are continually proportional*; C *being the center.* 12.

For draw MO perp. to MT, then (Prop. XI. Cor. 1.) $CA^2 : CD^2 :: CP : PO :: CP^2 : CP \times PO$, and

Fig. 12. and drawing PQ parallel to AD, then $CA^2 : CD^2 :: CP^2 : CQ^2$, by ſimilar triangles; whence $CP^2 : CP \times PO :: CP^2 : CQ^2$; therefore $CP \times PO = CQ^2$; by Ax. 7. Proportion. But in the right angled triangle OMT, where MP is perp. to OT, $OP \times PT = PM^2$ (Geom. II. 20. Cor. 2.) $= CD^2 - CQ^2$ (Prop. XIV.) $= CD^2 - CP \times PO$, by what is juſt proved. Therefore $OP \times PT + OP \times CP = CD^2$; that is, $OP \times CT = CD^2$.

But $OP \times CT : CP \times CT :: OP : CP ::$ (Pr. XI. Cor. 1.) $CD^2 : CA^2$; that is, $CD^2 : CP \times CT :: CD^2 : CA^2$.

Therefore (Ax. 7. Proport.) $CP \times CT = CA^2$; or CP : CA : CT ∺.

Cor. 1. *If* C *be the center*, PM *an ordinate*, MT *a tangent*, MO *perp. to* MT; *then* OP, CD, CT, *are continually proportional, or* $OP \times CT = CD^2$.

Cor. 2. *If* C *be the center* PM *an ordinate*; *then* BP : PA :: BT : AT; *and therefore* BT *is harmonically divided in the points* P, A.

For ſince CT : CA :: CA : CP; therefore (Propor. XIII.) $CT + CA : CT - CA :: CA + CP : CA - CP$;
that is, BT : AT :: BP : AP.

PROP. XVII.

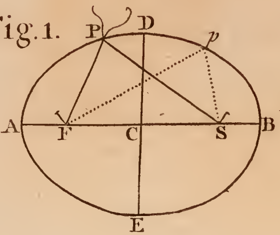
13. *If* MG *be an ordinate to the conjugate*, MV *a tangent at* M, C *the center*; *then* CG, CD, CV, *are continually proportional.*

By ſimilar triangles CV : VG :: CT : GM or CP :: (laſt Prop.) $CA^2 : CP^2$ or GM^2 (Prop. VII.) $:: CD^2 : DGE$ or $CD^2 - CG^2$.

And by diviſion $CV : CV - VG$ or $CG :: CD^2 : CG^2$; whence (Propor. 12.) $CV \times CG^2 = CG \times CD^2$ or $CV \times CG = CD^2$; therefore CG : CD : CV are ∺.

Cor.

Fig. 1.



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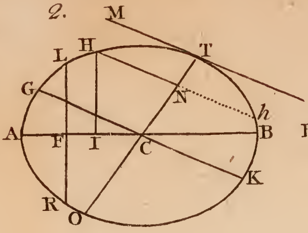

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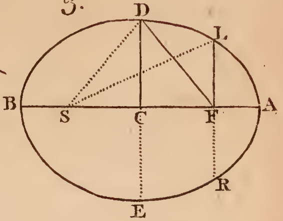

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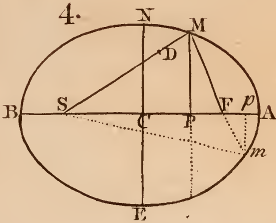

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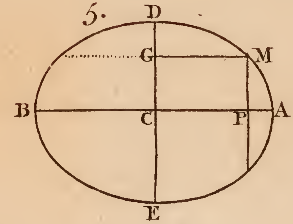

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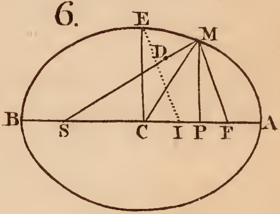

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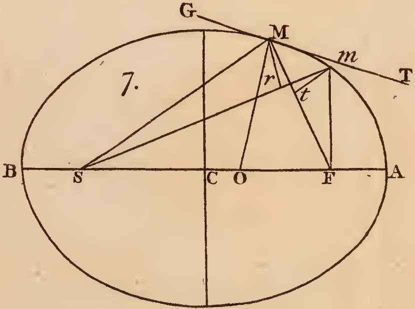

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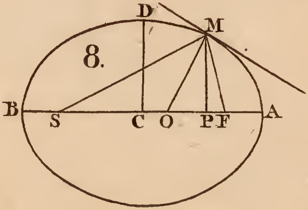

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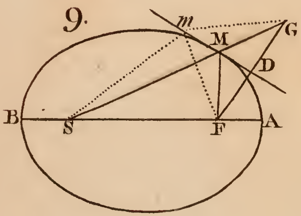

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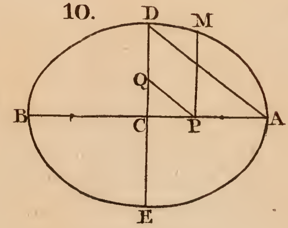

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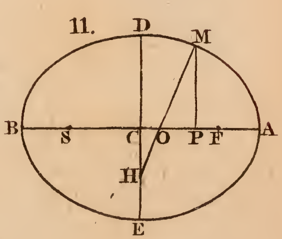


Conic Sections. Pl. I. *pa. 16.*

Cor. 1. *If* C *be the center*, MG *an ordinate to the conjugate*, MV *a tangent, and* MH *perp. to the tangent* MV; *then* HG, CA, CV, *are continual proportionals.* Fig. 13.

For $CG \times CV = CD^2$; but $CG \times CV : HG \times CV ::$ CG or PM : HG (by ſimilar triangles) :: OP : GM or CP (Prop. XI. Cor. 1.) :: $CD^2 : CA^2$; that is, $CD^2 : HG \times CV :: CD^2 : CA^2$; therefore (Ax. 7. Propor.) $HG \times CV = CA^2$.

Cor. 2. *If* C *be the center*, MG *an ordinate to the conjugate*, MH *perp. to the tangent* MV; *then* EG : GD :: EV : DV; *and* EV *is harmonically divided in the points* G, D.

For ſince CV : CD :: CD : CG, by compoſition and diviſion, $CV + CD : CV - CD :: CD + CG : CD - CG$. that is, EV : DV :: EG : DG.

PROP. XVIII.

If a ſemicircle AFD *be deſcribed on the tranſverſe or conjugate axis* AD *of an ellipſis* AKD, *and the ordinate* PM *be drawn cutting the circle in* I; *the tangents* MT, IT, *drawn from the points* M, I, *will interſect the axis in the ſame point* T. 14.

For (by Prop. XVI, and XVII.) the tangent MT will interſect the axis in T, ſo that CP, CA, CT, be in geometrical progreſſion. And (Geom. IV. 26.) If a tangent IT be drawn to touch the circle at I, it will interſect the diameter in T, ſo that CP, CA, CT be in geometrical progreſſion, the ſame as in the ellipſis.

Fig.

PROP. XIX.

15. *If on the tranſverſe axis* AB *as a diameter a circle* AKB *be deſcribed; and the ordinate* PM *be drawn cutting the circle in* N; *then* PN *will always be to* PM, *in the given ratio of the ſemitranſverſe* CA, *to the ſemiconjugate* CD.

For (Geom. IV. 17.) $PN^2 = BPA$; and (Prop. VI.) $CA^2 : CD^2 :: BPA$ or $PN^2 : PM^2$; and therefore $CA : CD :: PN : PM$.

Cor. *If a circle be deſcribed on the conjugate axis* DE *as a diameter, and an ordinate* QR *be drawn cutting the circle in* I; *then* QI *will always be to* QR, *in the given ratio of the ſemiconjugate* CD *to the ſemitranſverſe* CB.

For (Geom. IV. 17.) $QI^2 = DQE$. And (Prop. VII.) $CD^2 : CB^2 :: DQE$ or $QI^2 : QR^2$. Therefore $DC : CB :: QI : QR$.

PROP. XX.

16. *If the line* FD *be drawn from the focus, perpendicular to any tangent* MD; *then the interſection* D *will be in the circumference of the circle* ADB, *deſcribed on the diameter* BA *the tranſverſe axis.*

For produce SM from the other focus, and FD till they interſect in G. Draw CD, FM. Then in the right angled triangles FMD, GMD. The angle FMD (by Prop. XIII. Cor.) = GMD, and MD being common; therefore FD = DG, and FM = MG; and ſince FC = CS; therefore CD is parallel to SG (Geom. II. 12. Cor. 1.). Therefore the triangles FCD and FSG are ſimilar; therefore, ſince $FD = \frac{1}{2}FG$, $CD = \frac{1}{2}SG = \frac{SM+MG}{2}$

=

$= \frac{SM+MF}{2} = \frac{1}{2}BA = CA$; therefore D is in the circumference ADB. Fig. 16.

And by the ſame reaſoning, if S is the other focus, and SH be perp. to MH; the point H is in the circumference of the ſame circle ADB.

Cor. 1. *If a circle be deſcribed on the tranſverſe* AB, *and a tangent at* M, *cuts it in* D, H; *and the lines* FD, SH, *be drawn from the foci, to the points* D, H; *then* FD, SH *are perp. to the tangent* HMD.

Cor. 2. *If* F, S *be the foci,* C *the center,* MD *a tangent at* M; *then if* CD *be drawn parallel to* SM, *to cut the tangent in* D; *then* CD = CA *half the tranſverſe.*

Both theſe corollaries follow from the demonſtration of this Prop.

Cor. 3. *If a tangent* HD *interſects the circle* AHB *in* D *and* H; *the perpendiculars to it,* DF, HS, *will paſs thro' the foci* F *and* S.

Cor. 4. *If* CI *be drawn parallel to the tangent at* M; *then* IM = CA.

For then IM = CD, and CD = CA, by Cor. 2.

PROP. XXI.

If S, F *be the foci,* HT *a tangent at* M; *if* SH, FD *be drawn perp. to the tangent* MT; *then* SH×FD = BS×SA. 17.

Deſcribe a circle ARH about the tranſverſe, and produce HS to cut the circle at R; and draw CR. Then ſince ∠ H is right, RHD (Geom. IV. 14.) is a ſemicircle = BHA; therefore RB = DA, and ∠ RCB = ∠ DCA. Therefore in the triangles SCR, FCD, the ſides SC, CR, are equal to FC, CD, reſpectively, and the included angles at C equal; therefore SR = DF. (But Geom. IV. 20. Cor. 2.) HS×SR = BS×SA; that is, HS×DF = BS×SA = BF×FA.

Fig. 17. Cor. *The rectangle* HS×FD = *ſquare of the ſemiconjugate.* This is plain by Prop. III.

PROP. XXII.

17. *If* HT *be a tangent*; FD, SH *perpendiculars on it, from the foci*; *then* HM : MD : : HT : TD.

For deſcribing the circle BHA, about the tranſverſe BA, and drawing SM, FM; then the triangles SMH, FMD are ſimilar, for the angles at H and D are right, and SMH = FMD (by Prop. X.); therefore HM : MD : : HS : DF. Alſo the triangles TSH and TFD are ſimilar; therefore HS : DF : : TH : TD; whence HM : MD : : TH : TD.

Cor. *Hence* HM : MD : : TC+CF : TC—CF. For HM : MD : : TH : TD : : (by ſimilar triangles) TS : TF : : TC+CS : TC—CF.

PROP. XXIII.

18. *If* HMD *be a tangent at* M, *and* SH, FD, *perpendiculars, on it from the foci* S, F, CE *the ſemiconjugate. Then the rectangle* HMD = SMF—CE^2.

For deſcribing the circle BHDA, and drawing the ordinate PM, and producing it to the circle at N, and V; we have (Geom. IV. 20. Cor. 2.), HMD $= NMV = \overline{NP+PM}\times\overline{NP-PM} = NP^2-PM^2$ $= BPA-PM^2$ (Geom. IV. 17.) $= BPA-CM^2+CP^2$; but $BPA = CA^2-CP^2$, and $BPA+CP^2 = CA^2$. Alſo $CM^2 = CA^2+CE^2-SMF$ (Prop. IX.); therefore $HMD = CA^2-CM^2 = CA^2-CA^2-CE^2+SMF = SMF-CE^2$.

PROP. XXIV.

If S, F *be the foci,* HMD *a tangent at* M; FD, SH *perpendiculars on the tangent* HD, CE *the ſemiconjugate. It will be* SM : MF : : CE^2 : FD^2; *and* MF : MS : : CE^2 : SH^2. 18.

For (by Prop. X.) the angle SMH = FMD; therefore the triangles SMH, FMD are ſimilar; whence SM : MF : : SH : FD : : SH×FD : FD^2 : : (Prop. XXI. Cor.) CE^2 : FD^2.

And FM : MS : : FD : SH : : FD×SH or CE^2 : SH^2.

PROP. XXV.

If MP *be an ordinate to the tranſverſe,* HMT *a tangent at* M, C *the center,* AK, CG, BH *perp. to* AB. *Then* AK : PM : : CG : BH. 19.

For (Prop. XVI.) TC : CA : : CA : CP. And by compounding, TC : TC+CA or TB : : CA : CA+CP or BP. And (Propor. XIII.) TC : TB : : TC—CA : TB—BP; that is, TC : TB : : TA : TP.

But all the triangles TAK, TPM, TCG, TBH are ſimilar; therefore TA : TP : : AK : PM, and TC : TB : : CG : BH; whence AK : PM : : CG : BH.

Cor. TA : TP : : TC : TB.

PROP. XXVI.

If PM *be an ordinate,* C *the center,* AK, CG, BH *perpendicular to the tranſverſe* BA; HMT *a tangent at* M; *then* AK×BH = PM×CG = CD^2, *the ſquare of the ſemiconjugate.* 19.

For (Prop. XXV.) AK×BH = PM×CG; and (Prop. XVII.) PM : CD : : CD : CG; whence PM ×CG = CD^2.

Fig. Cor. 1. $AK\times BH = PM\times CG = \frac{1}{4}BA \times$ *latus*
19. *rectum* $= BSA$, S *being the focus.*

This follows from Prop. IV. Cor. 1. and Prop. III.

Cor. 2. *Lines drawn from* H *and* K *to the focus, make a right angle* HSK.

For by Cor. 1. $HB : BS :: SA : AK$; therefore (Geom. II. 16.) the triangles HBS, SAK are ſimilar, and $\angle BHS = ASK$.

But $HSK = HSA - ASK = HSA - BHS = HBA$ a right angle.

Cor. 3. *Hence a circle deſcribed on the diameter* HK *will paſs thro' the foci* S, F.

PROP. XXVII.

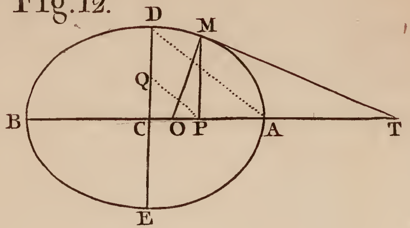
20. *If* S, F *be the foci,* HMD *a tangent at* M, MO *perpendicular to it, and* OG *perp. to* SM; *then* MG *is half the latus rectum.*

For draw SH, FD perp. to HD; then (by Prop. XXI. Cor.) $SH\times FD = CE^2$, the ſquare of the ſemiconjugate. And by the ſimilar triangles SMH, FMD, MOG, $MS : SH :: MO : MG$,
and $MF : FD :: MO : MG$;
and multiplying, $MF\times MS :: SH\times FD :: MO^2 : MG^2$.
or (XXI. Cor.) $MF\times MS : MO^2 :: CE^2 : MG^2$.
But (XII. Cor. 1.) $MF\times MS : MO^2 :: CA^2 : CE^2$.
therefore $CA^2 : CE^2 :: CE^2 : MG^2$
or $CA : CE :: CE : MG = \frac{1}{2}$ latus rectum,
by Prop. IV. Cor. 1.

Cor. 1. *If* MO *be perp. to the tangent at* M; L $=$ *latus rectum of the tranſverſe axis; then* $AB - L : L ::$ *rectangle* $SOF : MO^2$.

For we had before $SMF : MO^2 :: CA^2 : CE^2$ or $CA\times\frac{1}{2}L :: CA : \frac{1}{2}L$; and by diviſion $SMF - MO^2 : MO^2 :: CA - \frac{1}{2}L : \frac{1}{2}L :: BA - L : L$; but (Prop. XII.) $SMF - MO^2 = SOF$.

Fig. 12.



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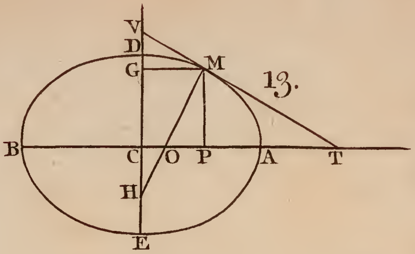

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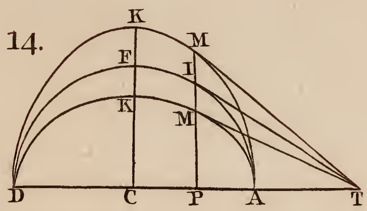

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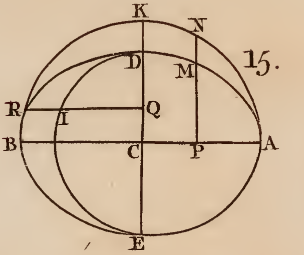

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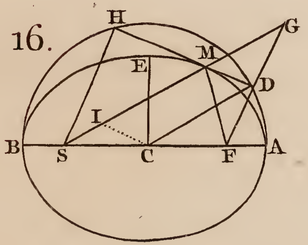

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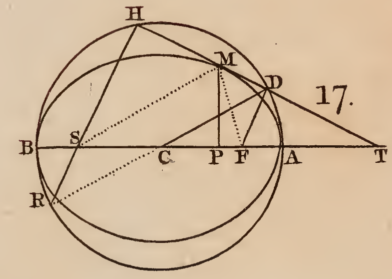

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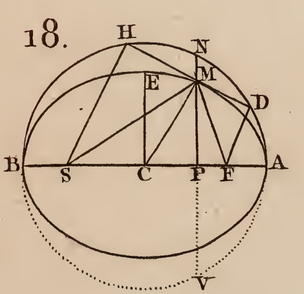

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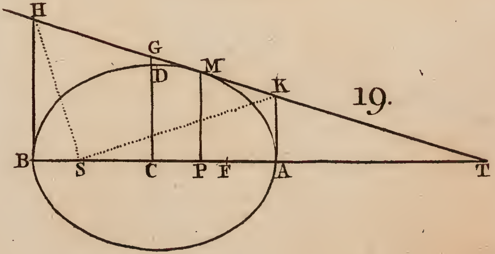


Cor. 2. *If* MH *be perp. to the curve at* M, *cutting the two axes in* O *and* H; *and* S, F *be the foci.* *Then* SOF $=$ MOH. Fig. 11.

For SOF : MO^2 : : BA—L : L : : (Prop. XV. Cor. 2.) HO : MO : : MO$\times$HO : MO^2; therefore SOF $=$ MOH.

Cor. 3. *Therefore a circle may be deſcribed from ſome point of the conjugate* CD; *which will paſs thro' the foci,* S, F, *and alſo thro'* M *and* H.

PROP. XXVIII.

If FH *be an ordinate at the focus,* TH *a tangent at* H, PM *any other ordinate, continued to the tangent at* G; *then this line* PG $=$ FM, *the line drawn from the focus to the top of the ordinate.* 21.

1. For FH is $\frac{1}{2}$ the latus rectum; and (XXVI. Cor. 1.) FH$\times$CN $= \frac{1}{2}$ latus rectum $\times$ CA $=$ FH $\times$CA; whence CN $=$ CA.

2. Draw AL, BZ, perp. to AB; then (Prop. XVI.) CA : CF : : CT : CA. And by diviſion (CA—CF) AF : CF : : (CF—CA) TA : CA; and compounding, AF : (AF$+$FC) AC : : AT : (AT $+$AC) TC : : (ſimilar triangles) AL : CN or AC; therefore AF $=$ AL.

3. Likewiſe (XVI. Cor. 2.) AF : FB : : TA : TB : : (ſimilar triangles) AL or AF : BZ; therefore BZ $=$ BF.

4. By Prop. XVI, $CA^2 = CF\times CT = CF \times \overline{CF+FT} = CF^2+CFT$, and $CA^2—CF^2$ or $CD^2 = CF\times FT$.

5. By ſimilar triangles, TF : FH : : TC : CN or CA : : TC$\times$CF or CA^2 : CA$\times$CF : : CA : CF; that is, TF : FH : : CA : CF.

6. By the laſt art. CA : CF : : TF : FH : : (ſimilar triangles) TP : PG.

And

Fig. 21. And (Prop. XVI. and VIII. Cor. 2.) CA : CF : : $\left\{\begin{array}{l} CT : CA \\ CP : FM - CA \end{array}\right\}$: : (compounding) CT+CP or TP : FM; therefore TP : PG : : TP : FM, whence PG = FM.

Hence are drawn the following corollaries,

Cor. 1. *If* TZ *be the focal tangent*; AL, CN, BZ *perpendiculars on* AB; *then* CN = CA.

Cor. 2. AL = AF, and BZ = BF.

Cor. 3. CF×FT = CD^2.

Cor. 4. TF : FH : : CA : CF.

Cor. 5. TA : AF : : TP : FM.
For by ſimilar triangles, TA : (AL) AF : : TP : (PG) FM.

Cor. 6. CA : CF : : TP : FM.
For CA : CF : : TF : FH : : TP : PG or FM.

PROP. XXIX.

22. *If* AB *be the tranſverſe axis*, C *the center*, F *the focus*; *and if* CF : CA : : CA : CT; *and* TE *be perp. to* TB. *Then if from any point* M, MF *be drawn to the focus, and* ME *parallel to* TB; *it will be* CF : CA : : FM : ME.

For if the ordinate FH be drawn, and the tangent TH, then (Prop. XVI.) it will be CF : CA : : CA : CT. And therefore (Prop. XXVIII. Cor. 6.) CF : CA : : FM : FP, or CF : CA : : FM : EM.

Cor. MF *is to* ME *always in a given ratio, whereever the point* M *is taken.*

SCHOLIUM.

The line TE is by the writers on conic ſections, called the *Directrix*.

PROP.

Fig.

PROP. XXX.

If BA, DE, *be the tranſverſe and conjugate axes; and if the difference between the ſemitranſverſe and ſemiconjugate, be ſet from the point* G *in the conjugate, to* I *in the tranſverſe, and continued to* M, *till* IM = *ſemiconjugate* CD. *Then* M *will be in the curve of the ellipſis.* 23.

For draw MP perpen. to AB, and MR parallel to it, then the triangles GRM and MPI are ſimilar; therefore GM : RM : : IM : IP; that is, CA : CP : : IM : IP; and $CA^2 : CP^2 :: IM^2 : IP^2$. And by diviſion $CA^2 : CA^2 - CP^2$ or BPA : : IM^2 or $CD^2 : IM^2 - IP^2$.

But $IM^2 - IP^2 = PM^2$. Therefore $CA^2 : CD^2 ::$ BPA : PM^2; which is the property of the ellipſis by Prop. VI.

Cor. *If from any point* M *in the ellipſis,* MI = CD *(the ſemiconjugate) be drawn to the tranſverſe, and continued to the conjugate at* G; *then* MG *is equal to the ſemitranſverſe* CA.

PROP. XXXI.

If two circles BDA, KLV *be deſcribed about the tranſverſe and conjugate diameters; and* CD *be drawn, and from* D, DP *perpendicular to* BA; *and from* L, LI *perp. to* DP; *then the point* I *will be in the ellipſis.* 24.

For ſince LI is parallel to CA, CD : CL : : DP : IP (Geom. II. 12. Cor. 4.); that is, CA : CG : : DP : IP, which is the property of the ellipſis by Prop. XIX.

PROP. XXXII.

25. *Any diameter* MN *is biſſected in the center* C.

From M draw the ordinate MP on the tranſverſe AB; take $CQ = CP$, and erect QN perp. to AB to cut MN in N. Then the triangles CPM and CQN are ſimilar, for all the angles are equal, alſo $CQ = CP$; therefore $CN = CM$.

PROP. XXXIII.

26. *If* ML, VR *be two conjugate diameters, and from the ends thereof, there be let fall the perpendiculars* MP, VQ *ordinates, to the tranſverſe axis* AB; *then* $CQ^2 = AP \times PB$.

For (Prop. XVI.) $CP : CA :: CA : CT$.

And $CP : (CA - CP$ or$)$ $PA :: CA : (CT - CA$ or$)$ $AT ::$ (Propor. XIII.) $CP + CA$ or $BP : PA + AT$ or PT; whence $CP \times PT = AP \times PB = AC^2 - CP^2$. And $AC^2 = CP \times PT + CP^2$.

Alſo, ſince CV is parallel to the tangent MT (def. 7.), the triangles TPM and CQV are ſimilar. And $AQB = AC^2 - CQ^2 = CP \times PT + CP^2 - CQ^2$, and (Prop. VI. Cor. 3.) $APB : AQB :: PM^2 : QV^2$ (ſimilar triangles) $:: TP^2 : CQ^2$; that is, $CP \times PT : CP \times PT + CP^2 - CQ^2 :: TP^2 : CQ^2$, and by permutation $CP \times PT : TP^2 :: CP \times PT + CP^2 - CQ^2 : CQ^2$; and by adding, $CP \times PT + TP^2 :: TP^2 :: CP \times PT + CP^2 : CQ^2$; and by permutation, $\overline{CP + PT} \times PT : \overline{CP + PT} \times CP :: TP^2 : CQ^2$, or $TP : CP :: TP^2 : CQ^2$, whence $TP \times CQ^2 = CP \times TP^2$, or $CP \times TP = CQ^2$, or $AP \times PB = CQ^2$.

Cor. 1. $AQ \times QB = CP^2$.

For $CP^2 = AC^2 - CPT = CA^2 - CQ^2 = \overline{CA + CQ} \times \overline{CA - CQ} = AQ \times QB$.

Cor. 2. $CP^2+CQ^2 = APB+AQB = CA^2$. Fig. 26.

For $APB = CQ^2 = AC^2-CP^2$,
and $AQB = CP^2 = AC^2-CQ^2$, and adding,
$APB+AQB = CP^2+CQ^2 = 2AC^2-CP^2-CQ^2$
and $2CP^2+2CQ^2 = 2AC^2$
or $CP^2+CQ^2 = AC^2$.

Cor. 3. *If* ML, VR *be two conjugate diameters,* MP, VQ *ordinates to the tranſverſe axis; then* $PM^2+QV^2 = CD^2$, *the ſquare of the conjugate.*

For $CA^2 : CD^2 :: BPA$ or $CQ^2 : PM^2 :: BQA$ or $CP^2 : QV^2$,
and $CA^2 : CD^2 :: CQ^2+CP^2 : PM^2+QV^2$, and alternately
$CA^2 : CQ^2+CP^2$ or $CA^2 :: CD^2 : PM^2+QV^2$,
therefore (Propor. 2.) $PM^2+QV^2 = CD^2$.

Cor. 4. $CA : CD :: CP : QV :: CQ : PM$.
For $CA^2 : CD^2 :: BPA$ or $CQ^2 : PM^2$.
and $CA^2 : CD^2 :: BQA$ or $CP^2 : QV^2$;
whence $CA : CD :: CQ : PM :: CP : QV$.

PROP. XXXIV.

The ſum of the ſquares of two conjugate diameters, is equal to the ſum of the ſquares of the two axes: $ML^2+VR^2 = AB^2+DE^2$. 26.

For $CM^2 = CP^2+PM^2$, and $CV^2 = CQ^2+QV^2$; whence $CM^2+CV^2 = CP^2+CQ^2+PM^2+QV^2 =$ (Pr. XXXIII. Cor. 2.) $CA^2+PM^2+QV^2 =$ (ib. Cor. 3.) CA^2+CD^2.

PROP. XXXV.

If MN, KR *be two conjugate diameters; then the rectangle of the diſtances of the foci, from the vertex, of the diameter* MN *is equal to the ſquare of its ſemiconjugate* CK: $FM \times MS = CK^2$. 27.

For (Geom. II. 28.) $FM^2+MS^2 = 2MC^2+2FC^2$; but $FM+MS = 2AC$, which ſquared is FM^2+

Fig. 27. $FM^2+MS^2+2FM\times MS = 4AC^2$, from which ſubtracting the firſt equation, $2FM\times MS = 4AC^2-2MC^2-2FC^2$, or $FM\times MS = 2AC^2-MC^2-FC^2$. But $AC^2-FC^2 = CD^2$, whence $FMS = AC^2+CD^2-MC^2$; but (Prop. XXXIV.) $AC^2+CD^2 = CM^2+CK^2$, therefore $FMS = CK^2$.

Shorter thus,

By Prop. IX. $CM^2 = CA^2+CD^2-SMF$, therefore $SMF = CA^2+CD^2-CM^2$; but (Prop. XXXIV.) $AC^2+CD^2 = CM^2+CK^2$, and $AC^2+CD^2-CM^2 = CK^2 = SMF$.

Cor. 1. *If* F, S, *be the foci, then* $FM\times SM = MC\times\frac{1}{2}$ *latus rectum of the diameter* MC.

For $KR^2 = MN\times$ latus rectum of MN (Prop. IV. Schol.), and dividing by 4, $KC^2 = MC\times\frac{1}{2}$ latus rectum $= SMF$.

Cor. 2. *If* MO *be perp. to the tangent at* M; *then* CA : CD :: CK : MO.

For (Prop. XII. Cor. 1.) $CA^2 : CD^2 :: SMF$ or $CK^2 : MO^2$, and CA : CD :: CK : MO.

PROP. XXXVI.

28. *If* F, S, *be the foci*; TH *a tangent at any point* M; FT, CI, SH *perpendiculars on the tangent*; MC, CK *ſemidiameters conjugate to one another*; *then* FM *is to* FT, *or* SM *to* SH, *as* CK *to* CD *the ſemiconjugate axis.*

For ſince $CF = CS$, $FT+SH = 2CI$. The triangles FMT and SMH are ſimilar; whence MF : FT :: MS : SH :: (Propor. XIII.) $MF+MS : FT+SH :: 2AC :: 2CI :: AC : CI$.

Again, AC : CI :: MF : FT
and AC : CI :: MS : SH by ſimilar triangles, there-

therefore $AC^2 : CI^2 :: MS \times MF : SH \times FT$, Fig.
but (Prop. XXXV.) $MS \times MF = CK^2$, and (Prop. 28.
XXI. Cor.) $SH \times FT = CD^2$, therefore $AC^2 : CI^2 ::$
$CK^2 : CD^2 :: MF^2 : FT^2$.
and $AC : CI :: CK : CD :: MF : FT :: MS : SH$.

Cor. $MF : FT :: AC : CI :: CK : CD$.
From the demonſtration.

PROP. XXXVII.

All circumſcribing parallelograms, whoſe ſides are 29.
parallel to two conjugate diameters, are equal to the reƈtangle of the two axes. GHNQ = RTXZ.

For let CI be perp. to GH which touches the ellipſis in M; then (Prop. XXXVI. Cor.) $BC : CI :: CK : CD$, therefore $CI \times CK = BC \times CD$; that is, MCKH = DCBT, and 4MCKH = 4DCBT; that is, GHNQ = RTXZ.

PROP. XXXVIII.

If BA *be the tranſverſe axis,* MV *any diameter;* 30.
AE, MT, *two tangents at* A *and* M, *interſeƈting* BA *and* VM *in* T *and* E; *then the triangle* CAE = CMT.

Draw the ordinates MP, AD, to CA and CM; then the triangles CPM, CAE are ſimilar, and likewiſe CDA, CMT, are ſimilar; whence $CD : CM :: CA : CT ::$ (Prop. XVI.) $CP : CA :: CM : CE$; therefore (Geom. II. 12. Cor. 1, 4.) the lines DP, MA, ET are all parallel to one another; whence (Geom. II. 10), triangle AME = triangle AMT; to which add the triangle CMA, then triangle CAE = triangle CMT.

Cor.

Fig. 30. Cor. 1. *The lines* DP, MA, ET *are parallel to one another.*

From the demonſtration.

Cor. 2. *The triangle* AIT = *triangle* MIE.

This appears by taking CAIM from the equal triangles CAE, and CMT.

Cor. 3. *Triangle* MPT = *triangle* ADE = *trapezoid* MPAE = *trapezoid* MDAT.

For ſince AME = AMT, add the triangle AMP, then PMT = MPAE, or add AMD (= AMP), then ADE = (MPT =) ADMT.

Cor. 4. *Triangle* CDA = *triangle* CPM.

For CD : CM : : CP : CA; whence CDA = CPM (Geom. II. 17. Cor. 1.)

PROP. XXXIX.

31. *If* BA *be the tranſverſe axis*, MV *a diameter*; MP, QL *ordinates*; AE, MT, *tangents*; QIF *perp. to* AB; *then the triangle* QIR = *trapezoid* IAEF.

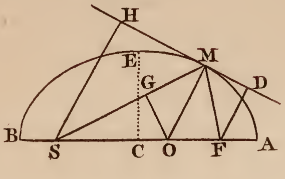
By ſimilar triangles CA : AE : : CP : PM : : CI : IF; and CA : AE : : (Propor. XIII.) CA+CP : AE+PM : : CA+CI : AE+IF.

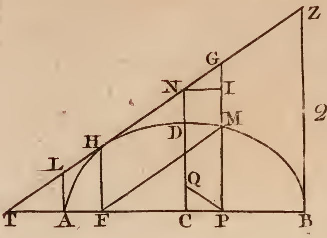
But rectangle BIA = $\text{AI} \times \overline{\text{CA}+\text{CI}}$, and rectangle BPA = $\text{AP} \times \overline{\text{CA}+\text{CP}}$; therefore BIA : BPA : : $\text{AI} \times \overline{\text{CA}+\text{CI}} : \text{AP} \times \overline{\text{CA}+\text{CP}} : : \text{AI} \times \frac{\text{AE}+\text{IF}}{2} : \text{AP} \times \frac{\text{AE}+\text{PM}}{2}$: : trapezoid AEIF : PAEM.

The triangles MPT, QIR are ſimilar, whence triangle MPT : QIR : : $\text{MP}^2 : \text{QI}^2$: : BPA : BIA : : trapezoid PAEM : trapezoid AEIF; but MPT = PAEM by Prop. XXXVIII. Cor. 3. therefore QIR = AEIF.

The

Fig. 20.





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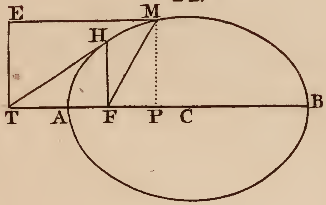

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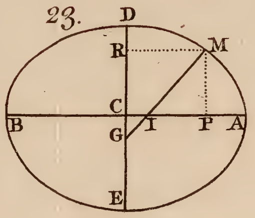

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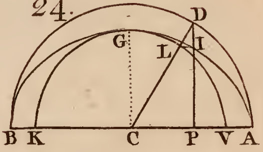

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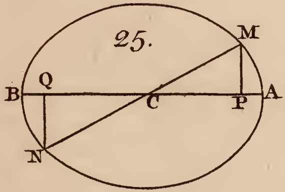

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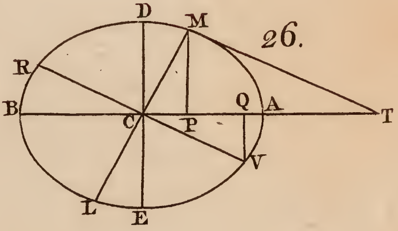

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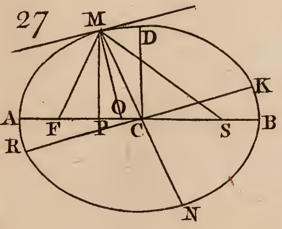

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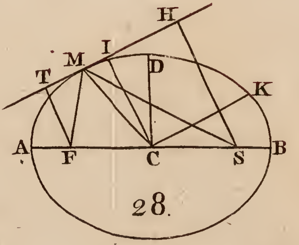

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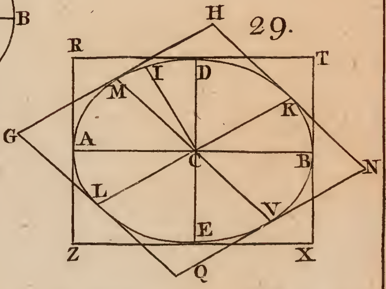


Pl. III. pa. 30.

The demonſtration is the ſame thing for the point Fig. q, uſing the ſmall letters inſtead of the great ones. 31.

Cor. *The triangle* QLF = *the trapezoid* LMTR. For (Prop. XXXVIII.) CAE = CMT, ſubtract CLR from both, and then LRAE = LMTR, but LRAE = LRIF + FIAE = LRIF + RQI = LQF = LMTR.

PROP. XL.

Let CK *be the ſemiconjugate to* MC, LQ *an ordinate thereto*; *then* $CM^2 : CK^2 :: $ *rectangle* MLV : LQ *ſquare*. 31.

Draw the tranſverſe axis AB, and KO perp. to it. Then (Cor. Prop. XXXIX.) FQL = LMTR, and for the ſame reaſon, OKC = CMT. But triangle CMT : CLR :: $CM^2 : CL^2$, and CMT : CMT—CLR :: $CM^2 : CM^2 - CL^2$; that is, triangle CMT : trapezoid LMTR :: CM^2 : rectangle VLM. But the triangles OKC, FQL are ſimilar; therefore $CK^2 : LQ^2$ :: triangle OKC : triangle FQL; or $CK^2 : LQ^2$ :: triangle CMT : trapezoid LMTR :: CM^2 : rectangle VLM; or $CK^2 : CM^2 :: LQ^2$: rectangle VLM.

Cor. 1. *As the tranſverſe* MV : *to its latus rectum* :: *ſo the rectangle* MLV : *to ſquare of the ordinate* LQ^2.

For MV : latus rectum :: MV^2 : MV × latus rectum or $\overline{2CK^2}$ (Sch. Prop. IV.) :: rectangle MLV : LQ^2.

Cor. 2. *The rectangles of the ſegments of any diameter, are as the ſquares of their ordinates.*

Cor. 3. *The diameter biſſects all its double ordinates*; *and the ordinates are equal at equal diſtances from the center.*

For

Fig. 31. For the proportion being the ſame for the ordinates, at the ſame point, on each ſide; theſe ordinates muſt be equal. And ſince all the terms of the proportion remain the ſame at equal diſtances from the center; the ordinates will be equal at equal diſtances.

PROP. XLI.

32. 33. *If two right lines* GI, HP *be drawn parallel to two conjugate diameters* DK, MV, *to interſect in* R; *then* $CM^2 : CK^2 :: $ *rectangle* HRP : *rectangle* GRI.

For (Prop. XL.) $PN^2 :: IO^2 :: DNK : DOK$; and (fig. 32.) $PN^2 : PN^2 - IO^2 :: DNK : DNK - DOK$. But $PN^2 - IO^2 = PN^2 - RN^2 = PRH$. And $DNK - DOK = DN \times \overline{NO + OK} - \overline{DN + NO} \times OK = DN \times NO - NO \times OK = \overline{DN - OK} \times NO = GR \times NO = GRI$.

Alſo (fig. 33.) $PN^2 : IO^2 - PN^2 :: DNK : DOK - DNK$. But $IO^2 - PN^2 = RN^2 - PN^2 = PRH$; and $DOK - DNK = DO \times \overline{ON + NK} - \overline{DO + ON} \times NK = DO \times ON - ON \times NK = \overline{DO - NK} \times ON = \overline{DN - OK} \times ON = GR \times ON = GR \times RI$.

Therefore $PN^2 : PRH :: DNK : GRI$,
or $PRH : GRI :: PN^2 : DNK :: CM^2 : CD^2$.

Cor. *Hence if two other lines be drawn parallel to* HP, GI; *the rectangles of their ſegments will be as the rectangles* PRH *and* GRI *parallel to them.*

Fig.

PROP. XLII.

If a line as HG *interſect any diameter* AB *in* D, 34.
and a diameter FK *be drawn parallel to* HG; *then* 35.
the rectangle ADB : *rectangle* GDH : : *as* $CB^2 : CF^2$.

Draw the conjugate MV, and AI parallel to GH; and the triangles CDE, CAI, are ſimilar, whence CI : CA : : CE : $\frac{CA\times CE}{CI} = DC$, and CI : AI : : CE : $DE = \frac{AI\times CE}{CI}$; alſo (by Prop. XL.) $MC^2 : CF^2 : : CM^2 - CE^2 : EH^2 = CF^2 - \frac{CE^2\times CF^2}{CM^2}$. Hence (fig. 34.) rectangle $ADB = AC^2 - CD^2 = AC^2 - \frac{CA^2\times CE^2}{CI^2} = CA^2 \times \frac{CI^2 - CE^2}{CI^2}$. Again $CF^2 : CM^2 : : AI^2 : CM^2 - CI^2$, whence $CM^2\times AI^2 = CF^2\times CM^2 - CF^2\times CI^2$, and $CM^2\times AI^2 + CF^2\times CI^2 = CF^2\times CM^2$. But the rectangle $GDH = EH^2 - ED^2 = CF^2 - \frac{CE^2\times CF^2}{CM^2} - \frac{AI^2\times CE^2}{CI^2} = CF^2 - \frac{CE^2\times CF^2\times CI^2 + AI^2\times CE^2\times CM^2}{CM^2\times CI^2} = CF^2 - \frac{CE^2\times CF^2\times CM^2}{CM^2\times CI^2} = CF^2 - \frac{CE^2\times CF^2}{CI^2} = CF^2 \times \frac{CI^2 - CE^2}{CI^2}$: therefore rectangle ADB : GDH : : $CA^2 : CF^2$.

And in fig. 35, the demonſtration is the ſame, only $ADB = CD^2 - AC^2$, or $-ADB = AC^2 - CD^2$, and $-GDH = EH^2 - ED^2$; whence the ſame concluſion will follow.

PROP.

Fig.

PROP. XLIII

36. *If two lines* GH, LE *interſect one another in* D, *and the ſemidiameters* CF, CK *be drawn parallel to them*; *the rectangle* GDH *is to the rectangle* LDE : : $CF^2 : CK^2$.

Thro' D draw the diameter AB; then by the laſt Prop. rectangle GDH : rectangle ADB : : $CF^2 : CB^2$. And rectangle ADB : rectangle LDE : : $CB^2 : CK^2$. Therefore GDH : LDE : : $CF^2 : CK^2$.

37. Cor. 1. *If any line* AB *cut two parallels* GH, EF, *in* D *and* L; *the rectangle* ADB : *rectangle* GDH : : *rectangle* ALB : *rectangle* ELF.

For EF, GH, being parallel to the ſame diameter; theſe rectangles will be as the ſquares of the ſemidiameters parallel to theſe lines.

38. Cor. 2. *If two parallel lines* EF, GH, *cut other two parallels* MN AK; *their ſeveral rectangles will be in the ſame ratio*; ABK : GBH : : MDN : GDH : : ACK : ECF : : MLN : ELF.

For each pair will be ſeverally proportional to the ſquares of the ſemidiameters parallel to theſe lines.

PROP. XLIV.

39. *If* DM *be a tangent at* M, *and* DN *any line cutting the ellipſis in* O *and* N; *and if the ſemidiameters* CF, CK *be drawn parallel to* DM, DN; *then* $DM^2 : CD \times DN : : CF^2 : CK^2$.

Although this may be inferred from Prop. XLII, by ſupppoſing the points G, H, to coincide in M (fig. 35.); yet I ſhall give the demonſtration of it here in particular, in reſpect of the diameter DACB.

Thro' the center C draw DAB, and draw AI parallel to DM. Fig. 39.

By ſimilar triangles $CI^2 : AI^2 :: CM^2 : DM^2$,
and (Prop. XL.) $AI^2 : MIV :: CF^2 : CM^2$,

Therefore,

(Propor. XVII.) $CI^2 : MIV :: CF^2 : DM^2$,
Again by ſimilar triangles,
$CI^2 : CM^2 :: CA^2 : CD^2$, and
by diviſion, $CI^2 : CM^2 - CI^2 :: CA^2 : CD^2 - CA^2$.
that is, $CI^2 : MIV :: CA^2 : ADB$.
whence $CA^2 : ADB :: CF^2 : DM^2$.
But (Prop. XLII.) $CA^2 : ADB :: CK^2 : ODN$.
whence $CF^2 : CK^2 :: DM^2 : ODN$.

Cor. 1. *If* DM, DN *be two tangents, and* CF, CK *be drawn parallel thereto from the center* C; *then* DM : DN : : CF : CK. 40.

For drawing DCB; $DM^2 : ADB :: CF^2 : CB^2$, and $ADB : DN^2 :: CB^2 : CK^2$; whence $DM^2 : DN^2 :: CF^2 : CK^2$, and DM : DN : : CF : CK.

Cor. 2. *If two parallel tangents* DM, TG, *meet a third tangent* DNT; *then* DM : DN : : TG : TN. 41.

For if CK be drawn parallel to DT, and CF to DM and TG; it will be DM : DN : : FC : CK, and TG : TN : : CF : CK; therefore DM : DN : : TG : TN.

Cor. 3. *If* PQ *be parallel to the tangent* DT, *and* RS *to the two tangents* DM, TG; *then* $POQ : ROS :: DN^2 : DM^2 :: TN^2 : TG^2$. 42.

For drawing CK, CF parallel to PQ, RS, by Prop. XLIII. $CK^2 : CF^2 :: POQ : ROS$ (and Cor. 1.) $:: DN^2 : DM^2$ (Cor. 2.) $:: TN^2 : TG^2$.

Cor. 4. *If* M, N, G, *be the points of contact of three tangents*, MD, DT, TG; *and* MD, TG, OE *be parallel to the ſemidiameter* CF; *and* DT, HR 43.

Fig. 43. *parallel to the ſemidiameter* CK; *then* $CF^2 : CK^2 :: IOE : NO^2 :: GT^2 : NT^2 :: GR^2 : LRP :: MH^2 : PHL$.

All this appears from what was before demonſtrated.

PROP. XLV.

44. *If* DM, DN, *touch the ellipſis in* M *and* N, *and* MN *be drawn, and any line* RL *parallel to* DN, *cutting* DM, MN, *in* G *and* H; *then* GR, GH, GL *are continually proportional.*

For (Prop. XLIV. Cor. 4.) $RGL : GM^2 :: DN^2 : DM^2 ::$ (ſimilar triangles) $GH^2 : GM^2$; therefore $RGL = GH^2$, and $RG : GH :: GH : GL$.

Cor. *If* FP *be parallel to* DN, *and touches the ellipſis in* P, *and cuts* DM *in* O, *then* OP = OF.

For then GL will fall on FP, R and L upon P, G upon O, and H upon F; whence FO = GH, GR and GL = OP; whence $OP^2 = FO^2$, and OP = FO.

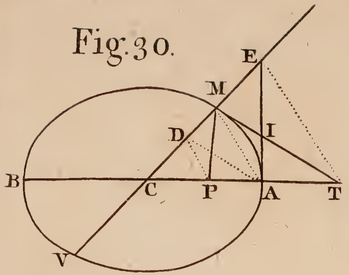
PROP. XLVI.

45. *If* MT, *a tangent in* M, *cut the diameter* BA *in* T, *and the ordinate* MP *be drawn*; *then* CP, CA, CT, *are continually proportional.*

For draw AK, BH tangents at A and B, to cut MT in K and H. Then ſince AK, PM, BH are parallel, $BP : AP :: HM : MK ::$ (Prop. XLIV. Cor. 2.) $HB : AK$ (ſimilar triangles) $:: TB : TA$. By diviſion $BP—AP : BP :: TB—TA : TB$; that is, $2PC : BP :: BA : TB$; and taking half of the antecedents $PC : PB :: BC : BT$. And by diviſion, $PC : PB—PC :: BC : BT—BC$; that is, $PC : BC :: BC : CT$.

Cor.

Fig.30.



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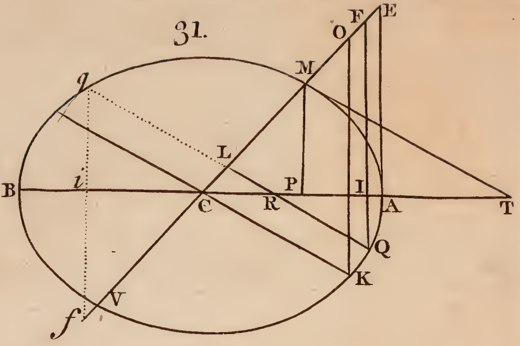

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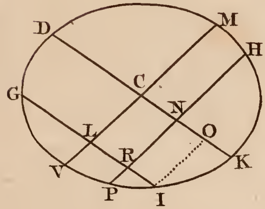

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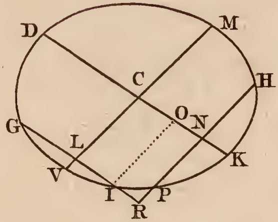

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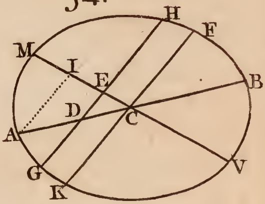

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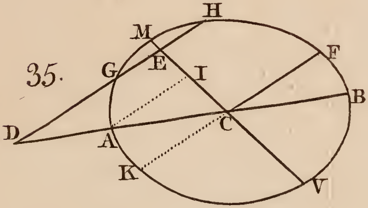

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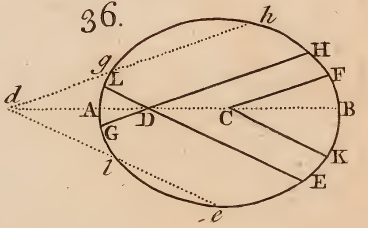

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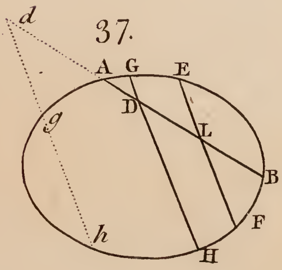

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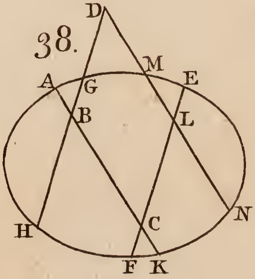

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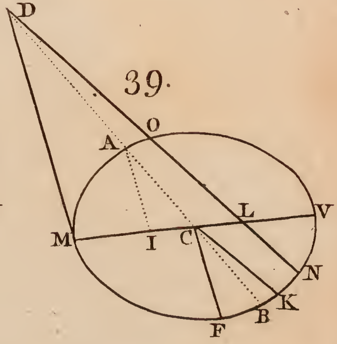


Pl.IV.*pa.*36.

Fig. 45. Cor. 1. *If* MT *be a tangent at* M, MP *an ordinate*; BP : PA : : TB : TA; *and therefore* BT *is harmonically divided in* A *and* P.

This appears from the demonſtration.

Cor. 2. *If* MT *be a tangent at* M, MP *an ordinate*; *then* PC : PA : : PB : PT.

For by the demonſtration, PC : PB : : BC : BT; and by diviſion PC : PB : : BC—PC : BT—PB : : AP : PT.

Cor. 3. *If* MT *be a tangent at* M, *and* MP *an ordinate*; *then* TA : TP : : TC : TB.

For ſince CT : CA : : CA : CP, CT—CA : CT : : CA—CP : CA; that is, AT : CT : : AP : AC; and (Propor. XIII.) AT : CT : : AT+AP : CT+AC, or AT : CT : : TP : BT.

Cor. 4. *Two tangents drawn from equal ordinates on each ſide the curve, meet in the diameter.*

PROP. XLVII.

46. *If* AB *be a diameter, and* AK, BH *two parallel tangents, at* A *and* B, *and* HT *a tangent at* M, *cutting the others in* K *and* H; MP *an ordinate,* CD *a ſemidiameter parallel thereto*; *then* AK, CD, BH *are in continual proportion.*

Continue CD to G. And by ſimilar triangles AK : PM : : TA : TP : : (Prop. XLVI. Cor. 3.) TC : TB : : (ſimilar triangles) CG : BH. Alſo ſince the tangent MG cuts CD in G; therefore (Prop. XLVI.) PM : CD : : CD : CG. Therefore (Propor. XVII.) AK : CD : : CD : BH.

Cor. 1. *If* AB *be any diameter,* CD *its ſemiconjugate,* PM *an ordinate*; AK, BH, HM, *tangents at* A, B, M; *then* $AK \times BH = PM \times CG = CD^2 = AB \times \frac{1}{4}$ *the latus rectum.*

Fig. For ſince AK : PM : : CG : BH ; therefore AK×
46. BH = PM×CG = CD^2 = (Schol. Prop. IV.)
$\frac{1}{4}$AB× latus rectum.

47. Cor. 2. *If the two parallel tangents* AK, BH, *be cut by the two tangents* KH, GR; *then* AK×BH = BG×AR.

For they are both equal to CD^2.

Cor. 3. AK : BG : : KO : OH.

For AK : BG : : AR : BH : : AR—AK : BH—BG : : KR : GH (ſimilar triangles) : : KO : OH.

Cor. 4. *A line drawn thro'* RH *wou'd interſect the line drawn thro'* KG, *in the diameter* AB *produced.*

Cor. 5. *If the tangent* GR *was to touch the oppoſite ſide* BDA, *as gr*; *it will ſtill be* AK×BH = B*g*×A*r*.

PROP. XLVIII.

48. *If two tangents* HK, HE, *touching the ellipſis in* D *and* R; *cut any diameter* AB *produced, in* K *and* E, *ſo that* AE = BK; *and if* FG *be a third tangent*; *then the rectangle* EF×GK = KH×RE, *a given quantity.*

Compleat the parallelogram KHEL, by drawing the lines EL, KL parallel to KH, HE, which will touch the ellipſis in O, P. Draw the diameter RP; then the figure RAC is ſimilar and equal to PBC, and REC to PKC, whence RE = PK. Then in reſpect of the diameter RP it will be (by Prop. XLVII. Cor. 3.) PK or RE : FR : : KG : GH; and by compounding, RE : FE : : KG : KH; whence KG×EF = RE×KH, a given quantity.

Cor. 1. VE×KZ = KH×RE *a given quantity.*

For by ſimilar triangles VE : EF : : GK : KZ, whence VE×KZ = EF×GK = KH×RE.

Cor. 2. *And if the third tangent* (FG) *touch the* Fig. *curve on the other ſide* (OP), *and interſect the tangents* 48. HK, HE *produced*; *the rectangle* EF×GK *will ſtill be the ſame.*

For the figure on the other ſide of KE, will be the ſame as VEKZ, and therefore that rectangle is the ſame as EV×KZ = KH×RE.

PROP. XLIX.

If two lines TF, TG, *touch the ellipſis in* H *and* 49. K, *and if* HK *be drawn, and biſſected in* O; *then if the line* TO *is drawn thro'* O, *it will biſſect all the lines* IN *within the ellipſis, which are parallel to* HK. *That is, if* HO = OK, *then* IP = PN.

Thro' the center C, draw FG parallel to HK; then ſince HO = OK, FC = CG. And ſince DE paſſes thro' C, it will be a diameter, and HO, OK ordinates parallel thereto; and IP, PN, being parallel thereto, will alſo be ordinates; therefore (Prop. XL. Cor. 3.) IP = PN.

Cor. 1. *If* LM *be drawn parallel to* HK; *the parts intercepted between the curve and tangent, on each ſide, are equal*; LI = MN.

For ſince HO = OK; therefore LP = PM, and IP being = PN, the remainder IL = NM. Likewiſe ſince CD = CE, therefore DF = EG, &c.

Cor. 2. *The tangent at* A *is biſſected in that point.*

Cor. 3. HO, OK, *as alſo* IP, PN *are ordinates to the diameter paſſing thro'* T, *the interſection of the tangents.*

PROP. L.

50. *If two lines* TL, TM, *touch the ellipſis in* H *and* K, *and* HK *be drawn thro' the points of contact; and from* T, *any line* TI *be drawn, cutting* HK *in* D, *and the curve in* F *and* I; *then* TF : TI : : DF : DI.

Thro' F, I, draw AC, LM parallel to HK, then (Prop. XLIX. Cor. 1.) LI = NM, and AF = BC, whence MI = LN, and CF = AB.

By ſimilar triangles AF : LI : : TF : TI.
and CF : MI : : TF : TI.
and multiplying, $AF \times CF : LI \times MI :: TF^2 : TI^2$.
that is, $AF \times AB : LI \times LN :: TF^2 : TI^2$.

But (Prop. XLIV. Cor. 3.) $AF \times AB : LI \times LN :: HA^2 : HL^2$: : (ſimilar triangles) $DF^2 : DI^2$; therefore $TF^2 : TI^2 :: DF^2 : DI^2$, and TF : TI : : DF : DI.

Cor. *If it be* TF : TI : : DF : DI; *the line joining the points of contact, (of the tangents* TH, TK,*) will paſs thro' the point* D.

PROP. LI.

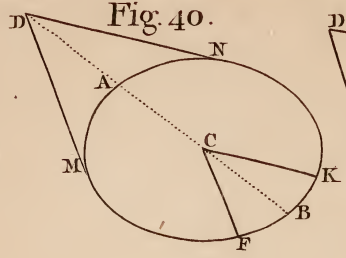
51. *If* MT, AH *be two tangents at* M *and* A; *and*
52. *if the ſemidiameters* CM, CA *be drawn, and produced to interſect the tangents in* H *and* T; *then the triangle* MNH = *triangle* ANT.

Draw PM parallel to AH; then by ſimilar triangles CM : CH : : CP : CA : : (Prop. XLVI.) CA : CT; and alternately CM : CA : : CH : CT; whence (Geom. II. 17. Cor. 1.) the triangle CAH = triangle CMT. Then ſubtracting (or adding) the quadrilateral figure CANM, the triangle MNH = ANT.

Cor. *The triangle* CAH = *triangle* CMT. From the demonſtration.

SCHO-

Fig. 40.



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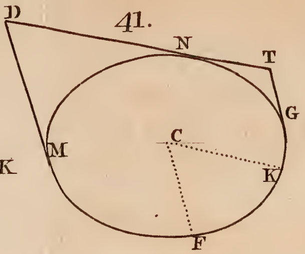

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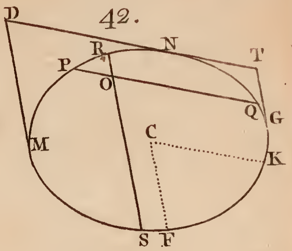

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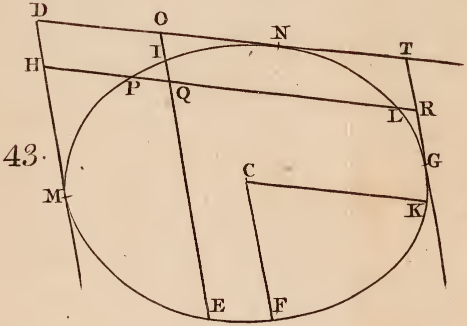

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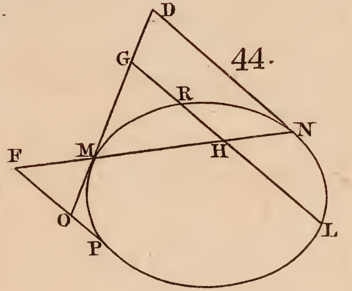

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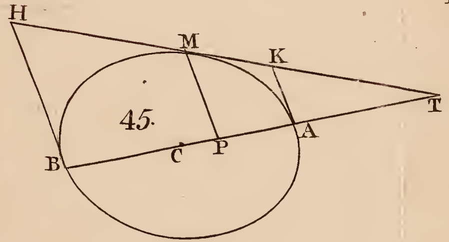

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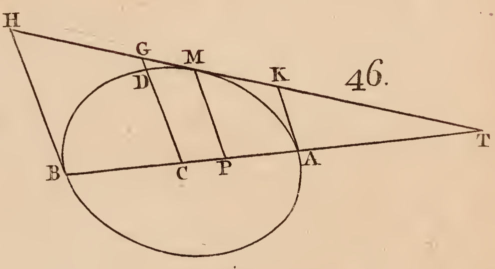

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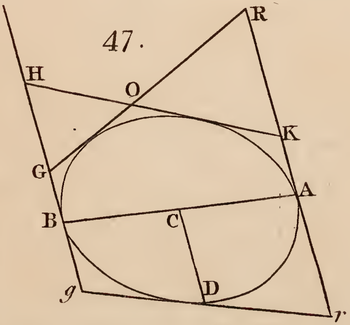

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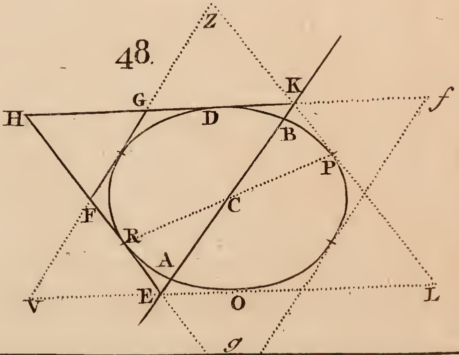


Pl.V. *pa*. 40.

Fig. 52.

SCHOLIUM.

What has been demonſtrated in Prop. XXXVIII, and Cor. 1. in regard to the tranſverſe axis; is here proved univerſally for any diameter whatever.

PROP. LII.

53. *If* ML, RV *be two conjugate diameters, and* MP, VQ *ordinates to any other diameter* AB, DE *the conjugate of* AB; *then* $CQ^2 = AP \times PB$.

This Prop. is demonſtrated from Prop. XLVI, and Prop. XL. Cor. 2. in the ſame manner as Prop. XXXIII. is from Prop. XIII. and Prop. VI. Cor. 3. And the ſame corollaries will follow, viz.

Cor. 1. $AQ \times QB = CP^2$.

Cor. 2. $CP^2 + CQ^2 = APB + AQB = CA^2$.

Cor. 3. $PM^2 + QV^2 = CD^2$.

Cor. 4. $CA : CD :: CP : QV :: CQ : PM$.

Cor. 5. *Hence alſo the triangle* CPM = CQV. For imagine PT = PC, then triangle PTM = PCM, and (Cor. 4.) CP or PT : QV :: CQ : PM; therefore (Geom. XVII. Cor. 1.) CQV = TPM = CPM.

PROP. LIII.

54. *If* ZH *be a tangent at any point* A; ML, VR *two conjugate diameters cutting the tangent in* Z *and* H; DE *the conjugate to* AB; *then the rectangle* $ZAH = CD^2$.

Draw the ordinates MP, VQ, to the diameter AB; then by ſimilar triangles CP : PM :: CA : AZ, whence $CP \times AZ = PM \times CA$ (Cor. 4. Prop. LII.) $= CQ \times CD$; therefore CP : CQ :: CD : AZ.

Again,

Fig. 54. Again, $CQ : QV :: CA : AH$, and $CQ \times AH = QV \times CA =$ (ib.) $CP \times CD$; therefore $CP : CQ :: AH : CD :: CD : AZ$; whence $AZ \times AH = CD^2$.

Cor. *If* SN *be drawn parallel to* CD; *then* $SF \times FG + FN^2 = CD^2$.

For $CA : AZ :: CF : FG$ (by ſimilar triangles).
and $CA : AH :: CF : FS$,
whence $CA^2 : AZ \times AH$ or $CD^2 :: CF^2 : FS \times FG$.
and dividing, $CA^2 : CD^2 :: CA^2 - CF^2$ or $AFB : CD^2 - SFG ::$ (Prop. XL.) $AFB : FN^2$. Therefore $CD^2 - SFG = FN^2$, and $FN^2 + SFG = CD^2$.

PROP. LIV.

55. *If* TP, TQ *be two tangents at* P, Q; *and* PQ *be drawn, and any line* LD, *cut* PQ *and the curve in* S, A, B; *then* $SL^2 : SD^2 ::$ *rectangle* ALB : *rectangle* BDA.

From the center C, draw CG, CM, CF parallel to TP, TQ, LD; and draw LI parallel to QT. Then the triangles SLI, SQD are ſimilar, as alſo PIL, PQT; whence $SD : SL :: DQ : LI$. But the ratio of DQ to LI is compounded of the ratio of DQ to PL, and (PL to LI or) PT to TQ; therefore $DQ : LI :: DQ \times PT : PL \times TQ$; therefore $SD : SL :: DQ \times PT : PL \times TQ$. Take the line E a mean proportional between AL and LB; and H a mean proportional between BD and DA; then $ALB = E^2$, and $ADB = H^2$.

Then (Prop. XLIV.) $GC^2 : FC^2 :: LP^2 : ALB$ or E^2. And $FC^2 : MC^2 :: BDA$ or $H^2 : DQ^2$. Therefore extracting the roots, $GC : FC :: LP : E$, and $FC : MC :: H : DQ$; therefore $GC : MC :: PL \times H : DQ \times E$; but (Prop. XLIV. Cor. 1.) $PT : QT :: GC : MC :: PL \times H : DQ \times E$, and multiplying means and extremes, $PT \times DQ \times E = QT \times PL \times H$.

whence H : E : : DQ×PT : PL×TQ : : SD : SL; Fig.
and H^2 or ADB : E^2 or ALB : : SD^2 : SL^2. 55.

Cor. *Hence in any circumſcribed trapezium* LNDO, *the diagonals* LD, NO, *interſect in the ſame point* S, *where the lines* PQ, KR, *interſect, that join the oppoſite points of contact.*

For if LD cuts the tangents PL, QD; we have, $SL^2 : SD^2$: : ALB or EE : ADB or HH; and SL : SD : : E : H, and SL+SD or DL : SL : : E+H : E.

Again, for the tangents KL and RD, it will be $SL^2 : SD^2$: : ALB : ADB; whence as before it will be, LD : SL : : E+H : E. Therefore PQ interſects LD in the ſame point S where KR interſects it; or the diagonal LD paſſes through the common interſection S, of the lines PQ, KR. And by the ſame reaſoning the diagonal NO paſſes thro' the ſame interſection S.

PROP. LV.

If GM *be any diameter,* GT, MD, *two tangents, at* G *and* M, DT *a tangent at* N; *then if* NC *be drawn, and its ſemiconjugate* CK; *the rectangle* DNT $= CK^2$. 56.

For draw the ſemiconjugate CF parallel to MD; then
(Prop. XLIV. Cor. 1.) CF : CK : : DM : DN
and CF : CK : : TG : TN; then
multiplying, $CF^2 : CK^2$: : DM×TG : DN×TN.
But (XLVII.) $CF^2 = DM \times TG$, therefore $CK^2 = DN \times NT$.

PROP.

PROP. LVI.

57. *If* AF, AG *be two tangents, and* FG *drawn thro' the points of contact. And thro'* A, *the interſection of the tangents,* AV *be drawn parallel to* GF. *Then if from any point in it as* V, *there be drawn* VL *thro'* O, *the middle of* FG, *to cut the ellipſis in* P *and* L. *Then will* $VP : VL :: OP : OL$.

From L draw AQRL, and from the interſection Q, draw QSP parallel to FG. Then (Prop. L.) $AQ : AL :: RQ : RL$, and $AL \times RQ = RL \times AQ$. The ratio of PQ to SQ is compounded of the ratio of PQ to OR and OR to SQ. But by ſimilar triangles, $PQ : OR :: LQ : LR$; alſo $OR : SQ :: AR : AQ$; therefore $PQ : SQ :: LQ \times AR : LR \times AQ$; and by diviſion, $PS : SQ :: LQ \times AR - LR \times AQ : LR \times AQ$. But $LQ \times AR - LR \times AQ = \overline{LR + QR} \times AR - LR \times AQ = QR \times AR + LR \times QR = AL \times QR = LR \times AQ$, by what went before; therefore $PS = SQ$; and conſequently, as QS is an ordinate (def. 9.), P is in the curve. Laſtly, by reaſon of the parallels FG, PQ, VA; $VP : VL :: AQ : AL ::$ (Prop. L.) $RQ : RL :: OP : OL$.

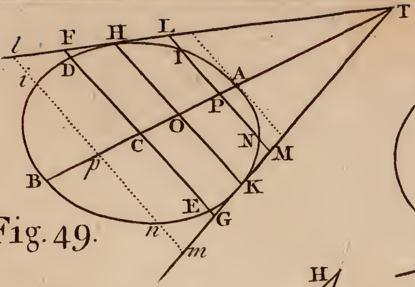
PROP. LVII.

58. *If* FA, GA *be two tangents at* F *and* G; *and if* GF *be drawn, and* AV *parallel to it. And from any point* V *in the line* AV; *let the tangents* VH, VI, *be drawn*; *then the line drawn thro' the points of contact*, HI, *will paſs thro' the point* O, *the middle of* FG.

Thro' O draw VL cutting the ellipſis in P and L; then in reſpect of the ordinate FG, it is (Prop. LVI.) $VP : VL :: OP : OL$. And in reſpect of the

the

Fig. 49.



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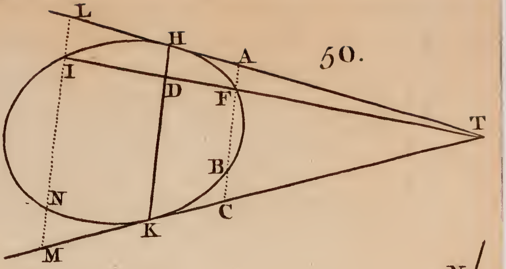

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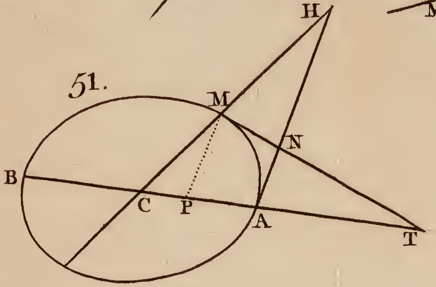

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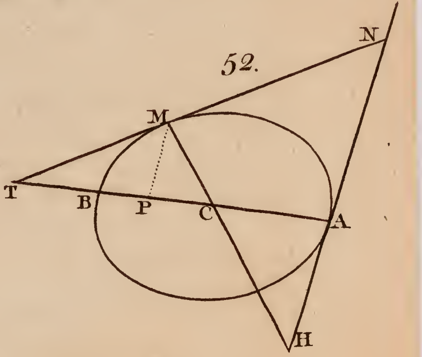

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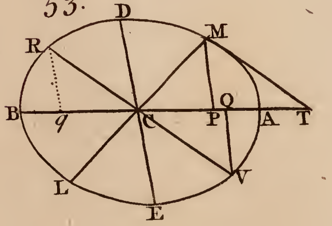

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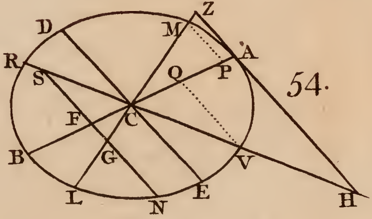

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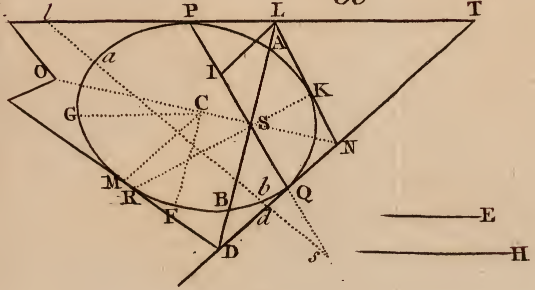

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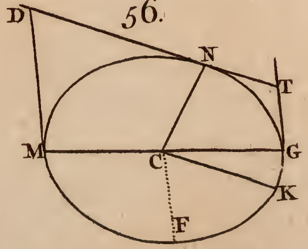

Pl. VI. *pa. 44.*

the line HI, it is (Prop. L.) VP : VL : : OP : OL; Fig. therefore O is common to both lines GF and HI; 58. therefore HI paſſes thro' O.

Cor. 1. *If* FA, GA *be any two tangents, and* FG *be drawn thro' the points of contact; and likewiſe* HV, IV, *two other tangents, and* HI *be drawn thro' their points of contact; draw* AV *thro' the concourſe of the tangents,* A, V. *Then* O, *the interſection of* FG, HI, *will be the point thro' which all lines paſs, which join the points of contact of any two tangents, drawn from any point in the line* AV.

Cor. 2. *If* GO *be an ordinate to any diameter* BE, GA *a tangent at* G, *meeting the diameter* BE *at* A; *and if* AV *be drawn parallel to* GO; *alſo through* O, *draw any line* HI; *then two tangents drawn from* H *and* I *will meet ſomewhere in the line* AV, *as at* V. Otherwiſe it could not be VP : VL : : OP : OL, as by Prop. LVI.

Cor. 3. *If* IH *be continued to* D, *and the tangents* DP, DL *drawn; then* LP *paſſing thro' the points of contact, will paſs thro'* O.

PROP. LVIII.

If VH, VI *be two tangents at* H, I; *and* IHD 58. *be drawn thro' the points of contact; and any line* VPOL *be drawn thro'* V; *and the tangents* PD, LD *be drawn; they will meet in the line* ID.

For thro' O (the interſection of VL and IH) draw FG, ſo that GO = OF; and draw VAD parallel to FG. Then (Prop. LVII. Cor. 2.) PD, LD, (tangents at P, L) will meet ſomewhere in the line VA, as at D. Then if DO, does not paſs thro' the points of contact H, I; let a line drawn from D to N paſs thro' the points of contact. Then (Prop. L.) we ſhall have VP : VL : : NP : NL. But by reaſon of

Fig. 58. of the tangents HV, IV, we have VP : VL : : OP : OL; therefore the former proportion cannot be true, except N and O coincide. Therefore DO paſſes thro' the points of contact, H, I; that is, the tangents PD, LD, meet in the line IH.

59. Cor. *If* FA, GA *be two tangents, and* GFB *be drawn thro' the points of contact; and from any point* B, *the tangents* BH, BI *are drawn. Then* IH *drawn thro' the points of contact, will paſs thro'* A.

For a line drawn thro' A will paſs thro' the points of contact H, I, and therefore no other can.

PROP. LIX.

60. *If any line* MN, *be drawn thro' the focus* F, *and the tangents* MR, NR, *drawn from* M *and* N; *then the line* RF, *drawn from the interſection of the tangents to the focus, will be perpendicular to the line* MN.

Make CF, CA, CT continual proportionals; at T erect TL perp. to the tranſverſe axis BAT. Then (by Prop. XVI.) a tangent drawn from the end of the ordinate erected at F, will cut the axis at T. And (by Prop. LVII. Cor. 2.), two tangents drawn from M and N will meet ſomewhere in the line TL as at R. Produce SM to G, ſo that SG may be equal to BA; draw GF which will be perp. to MR at D, by Prop. XIII. Draw ML parallel to BA.

The triangles SFG, MOG are ſimilar, whence SG or BA : SF : : MG : MO; that is, CA : CF : : MF : MO. But (Prop. XXIX.) CA : CF : : ML : MF; therefore ML : MF : : MF : MO, and ML$\times$MO $=$ MF^2. But the triangles MDO and MLR are ſimilar, being right angled at D and L; whence MD : MO : : ML : MR, therefore MD$\times$MR $=$ ML$\times$MO $=$ MF^2; and MD : MF : : MF : MR; therefore the triangles MDF, MFR are ſimilar, whence the angle MFR $= <$ MDF $=$ a right angle.

Cor. *If a circle be deſcribed about the diameter* MR, Fig.
it will paſs thro' the focus F, *and thro' the points* L, G. 60.

For MFR and MLR are right angles, and the triangle MGD being equal and ſimilar to MFD, is contained in the oppoſite ſemicircle.

LEMMA.

If EM, EN *be two lines meeting in* E, *and if it* 61.
be CA : CM : : DB : DN, *a given ratio, and* AB, 62.
CD, MN *be drawn. And if it be made* CA : DB : : AE : BG : : EP : GE. *Then if* OHK *be drawn thro' the middle of* PE, EG; *it will biſſect all the lines* AB, CD, MN, *&c.*

For take any of the lines, as MN; and make NF = GE, and draw MF. Make ES = EG, and draw PS, and OR, EI, HQ parallel thereto; alſo draw VEL parallel to OH, then GN = EF, RE = EH, EI = HV; but by the ſimilar triangles OEI, OQH; it is OE : OQ : : EI or HV : HQ; but ſince RE = EH, OE is = EQ, or OE = $\frac{1}{2}$OQ, therefore HV = $\frac{1}{2}$HQ, and HQ is biſſected in V. But ſince EM is to GN or EF, and alſo EO to ER, or EQ to EH, as AC is to BD; therefore the triangles EHQ, EFM are ſimilar, whence MF is parallel to HQ. Therefore ſince HQ is biſſected by EL, MF will be biſſected in L; therefore drawing LK parallel to EN; it will biſſect MN at K; and meet OHK at K, becauſe LK = EH = $\frac{1}{2}$EG = $\frac{1}{2}$FN. And thus any other line as CD is proved to be biſſected by HK.

Cor. 1. *Hence if a line biſſects two of the lines* AB, CD, MN, *&c. it biſſects all the reſt.*

Cor 2. *The line* VEL *is the locus of all the middle points of the lines* MF. *And* HOK *is the locus of the middle points of all the lines* AB, CD, MN, *&c.*

SCHO-

Fig. 61.

Scholium.

62. Any line MN might be divided in a given ratio, by the right line OHK; if inſtead of making EH = HG, and EO = OP, we make EH to HG, and EO to OP in the ratio of MK to KN as deſigned.

PROP. LX.

63. *If a trapezium be circumſcribed about an ellipſis, as* ABFE; *and if the diagonals* AF, BE, *be biſſected; the line drawn thro' the points of biſſection, will paſs thro' the center* C.

Thro' the points of contact of any two oppoſite ſides, AE, BF, draw the line HI, and the diameter GD parallel to it. Then (by Prop. XLVIII. and Cor. 2.) GA×DB = GE×DF, whence GA : DF : : GE : DB. Therefore if AF, GD, EB be drawn; then (by the Lemma Cor. 1.) the line that paſſes thro' the middle of AF, EB, paſſes thro' the middle of GD. But as GD is parallel to HI, the middle of GD is the center C.

64. Cor. *And if any four lines touch the ellipſis, as* AE, EF, FB *and* BA; *and the diagonals* EB, AF *be drawn to the oppoſite interſections. The line drawn thro' the middle points of theſe diagonals will paſs thro' the center.*

For draw HI thro' the points of contact of two oppoſite ſides EG, FB; and GD parallel to it thro' the center. Then (Prop. XLVIII.) GE×DF = GA×DB; and GE : DB : : GA : DF by diviſion : : EA : BF; therefore if GD, EB, AF be drawn, then (Cor. 1. Lem.) the line paſſing thro' the middle of AF, EB, paſſes thro' C the center.

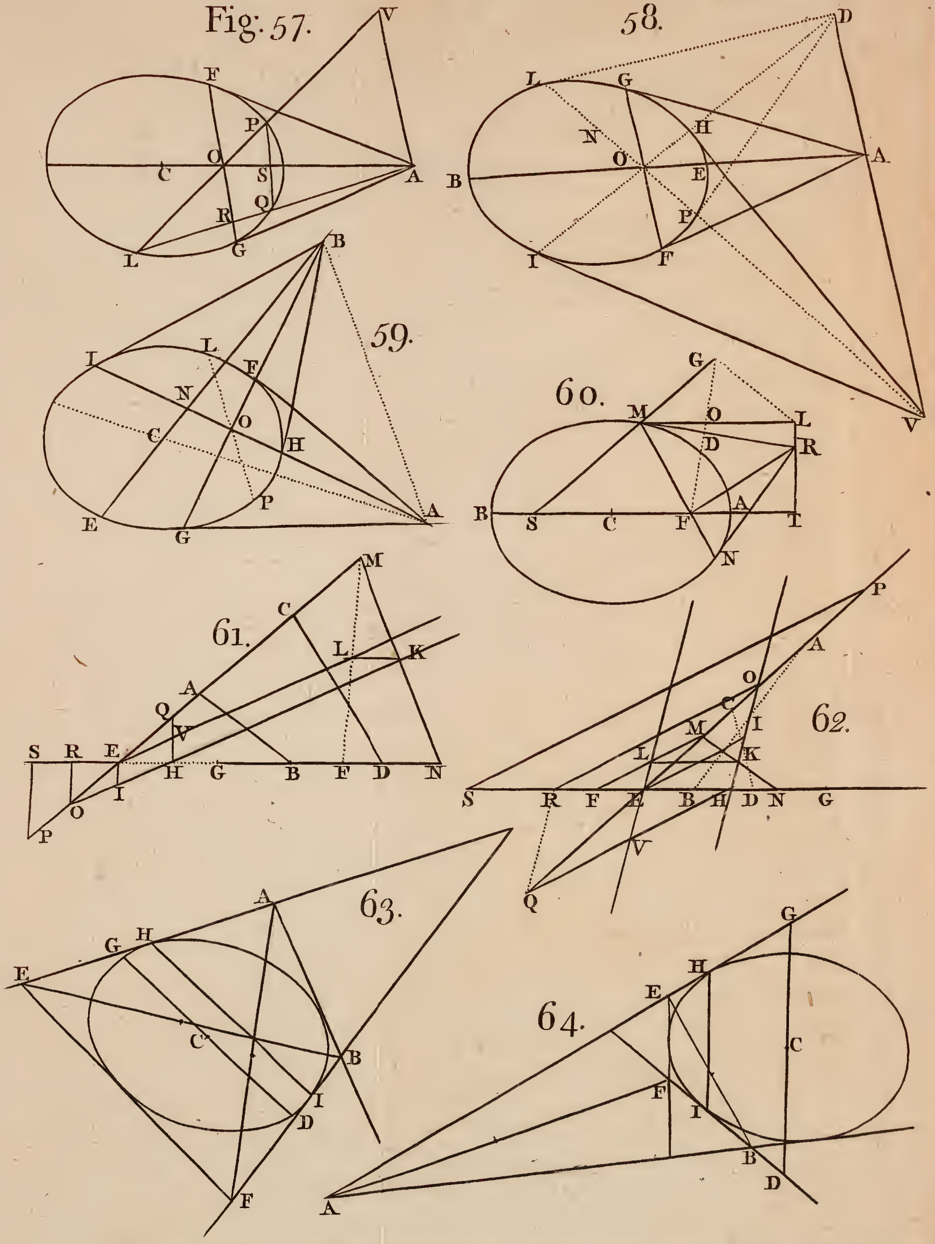
Fig: 57.
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Pl. VII. *pa.* 48.

PROP. LXI.

If a circle be deſcribed about the conjugate axis DE, *and* MG *be any right line*; *and if* MN, GQ *be drawn parallel to the tranſverſe axis* AB, *and* QP *parallel to* GM; *and if it be made, as* AC : CD : : MN : ON : : PN : IN : : GL : QL. *Then I ſay* GM *will be to* LO *in a given ratio, wherever* MG *be placed, in a parallel ſituation.* 65.

Produce MG, NQ to F; then ſince NM : NO : : QG : QL, OL produced will fall in F. And becauſe PQ is parallel to MG, it will be NF : NQ : : NM : NP : : (by conſtruction) NO : NI, or NF : NO : : NQ : NI; therefore (Geom. II. 12. Cor. 1.) FO (or LO) and QI are parallel, and OLQI is a parallelogram; therefore QI = LO; whence MG will be to OL, as QP to QI.

Cor. *If* M *(the end of* MG*) be in the curve of the ellipſis*; *then* O *(the end of* OL,*) will be in the periphery of the circle.*

This appears from Prop. XIX. Cor.

PROP. LXII.

If AB *be any diameter, and if thro' any point of the curve as* P, *two lines* APH, BPN, *be drawn, and alſo the ordinate* DF, *interſecting the lines in* E *and* G. *Then* DE, DF, DG *will always be in continual proportion.* 66.

Draw the conjugate IL, and PO parallel to it, and the tangents AN, BH. Then by ſimilar triangles,

AB : BH : : AO : OP
and BA : AN : : BO : OP

then

Fig. then by multiplying, $AB^2 : AN \times BH :: AO \times OB :$
66. $OP^2 ::$ (Prop. XL) $AB^2 : IL^2$; therefore $AN \times BH = IL^2$.

Again (Prop. XL.), $AB^2 : ADB :: IL^2 : DF^2$.
and by ſimilar triangles $AB : AD :: BH : DE$
and $AB : BD :: AN : DG$.
and multiplying, $AB^2 : ADB :: AN \times BH : DG \times DE$.
therefore $IL^2 : DF^2 :: AN \times BH : DG \times DE$.
But $IL^2 = AN \times BH$, therefore $DF^2 = DG \times DE$, whence $DE : DF :: DF : DG$.

Cor. *If from the ends of any diameter* AB, *two lines* AH, BN *be drawn, thro' any point* P, *of the curve, to interſect the tangents* AN, BH *in* N *and* H. *The rectangle of the tangents is equal to the ſquare of the conjugate* : $AN \times BH = IL^2$.

From the demonſtration.

PROP. LXIII.

67. *If* C *be the center,* F *the focus, and if* $CF : CA :: CA : CT$, *and the directrix* TO *be drawn perp. to* CA. *And if any line* MNO *be drawn; and from the focus* F, MFH, FN *and* FO *be drawn. Then* FO *biſſects the angle* NFH.

Draw MP, NQ perp. to TO, and ND parallel to MF. Then by ſimilar triangles, $MP : NQ :: MO : NO :: MF : ND$; and alternately $MP : MF :: NQ : ND$. But (Prop. XXIX.) $CA : CF :: MP : MF :: NQ : NF$; therefore $ND = NF$; therefore $\angle NFD = NDF = DFH$, by reaſon of the parallels ND, FH.

PROP.

PROP. LXIV.

If AB, ED *be the tranſverſe and conjugate axes, and if the diameters* GCK, *and* HCL *be drawn parallel to* AD, AE; *theſe diameters will be equal, and conjugate to one another.* 68.

For ſince EA = AD = DB = BE, ADBE is a parallelogram, and BE parallel to DA which is parallel to KG; therefore the angle EBA = GCA. And ſince BC = CA, therefore EF = FA; and conſequently GK is the diameter of EF, FA, and HL (parallel to EA) its conjugate. Likewiſe the $<$ GCA = EBA = DBA = LCA, conſequently CG = CL, or GK = HL.

Cor. 1. *There are no other conjugate diameters equal to one another but* GK, HL.

For GK grows greater in approaching to A, and its conjugate LH grows leſs towards E; or GK grows leſs towards E, and its conjugate LH grows greater towards A, and G and L cannot both approach to A.

Cor. 2. *The equal diameters* GK, LH *comprehend the leaſt angle of any two conjugates.*

For whilſt CL approaches to A, CG approaches to E, and the angle GCL continually increaſes, till L come to A, and G to E, where it becomes a right angle ECA.

PROP. LXV.

If AB *be any diameter,* BG *a tangent at* B, *and equal to the latus rectum; and if* AG *be drawn cutting the ordinate* PM *(produced) in* D. *Then* $PM^2 = BP \times PD$. 69.

By Cor. 1. Prop. XL. AB : BG : : APB : PM^2. But from the ſimilar triangles APD, ABG; AB :

Fig. 69. $BG :: AP : PD :: AP \times PB : PD \times PB$; therefore $APB : PM^2 :: APB : PD \times PB$; whence $PM^2 = PD \times PB$.

Cor. *The ſquare of the ordinate,* PM^2 *is leſs than the rectangle* $BP \times BG$ *(the rectangle under the parameter and abſciſſa), by the rectangle* $PB \times FG$, *which is ſimilar to the rectangle* $AB \times BG$. *And by reaſon of that defect, the curve is called an ELLIPSIS.* For if DF be parallel to AB, the rectangle DFG is ſimilar to ABG; becauſe from the ſimilar triangles ABG, DFG, it is $AB : BG :: DF : FG$.

SCHOLIUM.

The rectangle under AB the diameter, and BG its parameter, is called the *Figure* of that diameter.

PROP. LXVI.

70. *If* OT *be a tangent at* M; *and a line* MF *be drawn from the point of contact to the focus*; *and* CO *be drawn from the center parallel to* MF, *to interſect the tangent in* O. *Then* $CO = CB$ *half the tranſverſe axis.*

Draw the ordinate MP perp. to the axis BA. Then (Prop. VIII. Cor. 2.) $CF \times CP = CA^2 - CA \times FM$, and $CA \times FM = CA^2 - CF \times CP$. But (Prop. XVI.) $CA^2 = CP \times CT$; therefore $CA \times FM = CP \times CT - CP \times CF = TF \times CP$. But by ſimilar triangles, $TC : CO :: TF : FM :: TF \times CP$ or $CA \times FM : FM \times CP$; whence $CO \times CA \times FM = TC \times CP \times FM$, and $CO \times CA = CT \times CP = CA^2$, and $CO = CA$.

Cor. *If* MT *be a tangent*, F *the focus*, MP *an ordinate*; *then* $CA \times FM = TF \times CP$.

PROP. LXVII.

If F *be the focus,* C *the center,* QZ *any diameter,* 71.
MN *parallel to it drawn thro' the focus. Then* AB,
QZ, MN, *are in continual proportion.*

Draw the diameter CLD, thro' the middle of MN; and MP parallel to it, an ordinate to QZ; draw MO the tangent at M to cut CQ in O. Then (Prop. LXVI.) CO = CA. And (Prop. XVI.) CP, CQ, CO are in continual proportion; that is, LM, CQ, CA, are in continual proportion. And doubling the terms, MN, QZ, AB, are proportionals.

Cor. $CQ^2 = LM \times CA$.

PROP. LXVIII.

If any line GP *be drawn thro' the focus* F; *the* 72.
rectangle GFP $= \frac{1}{2}GP \times \frac{1}{2}$ *latus rectum.* 73.

Draw HK parallel to GP thro' the center C, and CD biſſecting GP in L. Then (XLII.) $CH^2 : GFP :: CB^2 : AFB ::$ (IV. Cor. 3.) $CB^2 : CB \times \frac{1}{2}$ latus rectum $:: CB : \frac{1}{2}L :: CB \times GL : \frac{1}{2}L \times GL$. But (LXVII. Cor.) $CH^2 = GL \times CB$; therefore GFP $= GL \times \frac{1}{2}L = \frac{1}{2}GP \times \frac{1}{2}$ latus rectum.

PROP. LXIX.

If QT, PT *be two tangents; and if* QF, PF, 74.
be drawn from the points of contact to the focus. Then 75.
the line TF *will biſſect the angle* QFP.

Make CF : CA : : CA : CG, and draw the directrix MGR perp. to BG. Produce TF to K, and draw LR, KR tangents at L, K (Prop. LVIII.

Fig. and Cor.). Let QD, PE, be perp. to MR. Then
75. (Prop. XXIX. Cor.) QF : PF : : QD : PE : : (ſimilar triangles) QR : PR : : (Prop. L.) QN : PN. Therefore (Geom. II. 25. Cor. 1.) FN biſſects the angle PFQ.

PROP. LXX.

76. *If a trapezium* ABDC *be inſcribed in an ellipſis; and any point* E *be taken in the curve, from which two lines* EN, EH, *are drawn parallel to any two of the adjoining ſides, as* AB, AC; *to interſect the oppoſite ſides in* N, Q; *and* H, R. *Then taking the rectangles on the oppoſite ſides;* ER×EH *will always be to* EN×EQ, *in a given ratio.*

Draw DFG, BPL parallel to AC; thro' L and C, draw OCLS. Thro' the middle of AC, BL, draw the diameter MK, which will biſſect the parallels DF, TE; and alſo SG, OR terminated at AB, CL. Then will SD = FG, OT = ER, which would alſo be ſo if SG, RH were tangents.

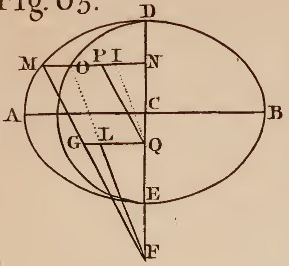
By ſimilar triangles, and the parallel lines; HO : SD : : OC : SC : : AR : AG; and alternately HO : AR or EQ : : SD or FG : AG.

Again, BP or ER : PN : : DG : GB; and by multiplying, HO×ER : EQ×PN : : FG×DG : AG ×GB : : (Prop. XLIII. Cor. 2.) ER×RT or ER× EO : RA×RB or EQ×EP. And by compounding, FG×DG : AG×GB : : HO×ER+EO×ER : EQ×PN+EQ×EP : : EH×ER : EQ×EN. But it is plain, as long as the points A, B, D, C, are fixed; that FGD to AGB is a given ratio; and therefore that of REH to QEN.

Cor. 1. *The ſame things ſuppoſed; it will be,* EH : EN : : EO : EP : : HO : PN.

For EH×ER : EQ×EN : : FG×GD : AG×GB : : HO×ER : EQ×PN; therefore EH×ER : HO×ER : : EQ×EN :

Fig. 65.



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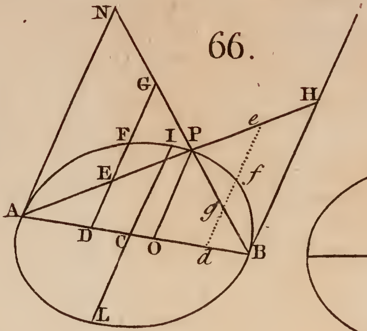

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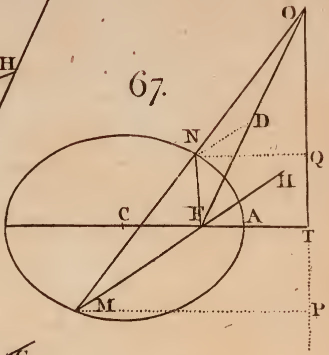

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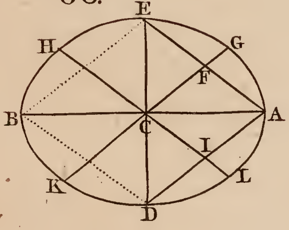

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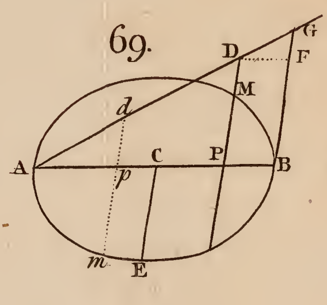

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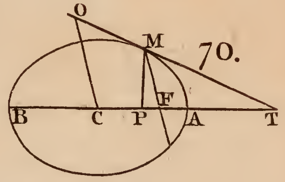

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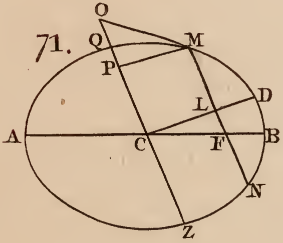

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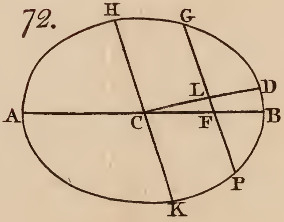

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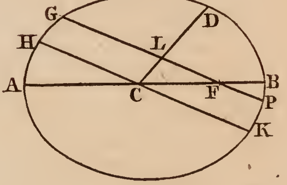

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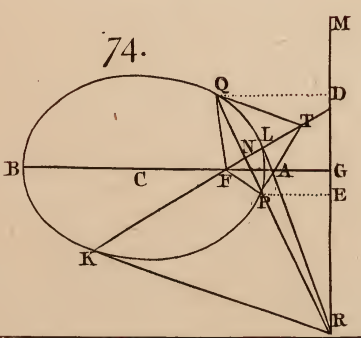

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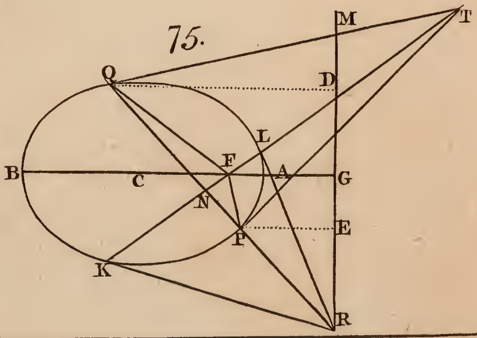


Pl.VIII. *pa.51.*

EQ×EN : EQ×PN ; and by equal diviſion EH : Fig.
HO : : EN : PN ; and by diviſion EH : EO : : 76.
EN : EP.

Cor. 2. *If the points* A, B, C, E, *be fixed, and* D *any how changed ; it will ſtill be* EH : EN ; *and* HO : PN, *in the given ratio of* EO *to* EP *or* RT *to* RB.

Cor. 3. *If the points* A, B, *coincide ;* AG *becomes a tangent ; and if* C *and* D *alſo coincide ;* DH *becomes a tangent, and then* Q, N *coincide.*

Cor. 4. *If* BC *be drawn to cut* EH *in* I. *And* 77.
if it be EH : EI : : EN EV ; *then if* BV *be drawn,*
it will touch the curve in B.

For (by Cor. 2.) If D be moveable, it will be EH : EN : : EO : EP, wherever D is. Therefore ſuppoſe D to move round, till it coincide with B, then BND will become a tangent at B, and N will fall upon V ; that is, BV will be a tangent.

Cor. 5. *If you make* EH : EN : : EO : EP ; *and* 78.
draw OC, BP, *they will meet at* V *in the curve.*

For ſuppoſing V a point in the curve, it will be (Cor. 2.) EH : EN : : EO : EP. Therefore V cannot be out of the curve.

Cor. 6. *If there be five points* B, D, V, C, E, *in the curve. Thro' any point* E, *draw* EPN, EOH, *cutting* BD, BV, DC, VC, *in* N, P, H, *and* O ; *ſo that* EH : EN : : EO : EP. *Then if* CA, BA *be drawn parallel to* EH, EN, *they will meet at* A *in the curve.*

This is the reverſe of the laſt Cor. according to which, if A were out of the curve, V would not be in it.

Cor. 7. *If* A, B, *coincide, then* AB *becomes a tangent. And ſo of any other two points.*

PROP. LXXI.

79. *If a trapezium* ABDC *be inſcribed in an ellipſis; and from any point* E *of the curve, four lines* EL, EK, EO, EM; *be drawn to the four ſides of the trapezium, in any angles whatever, equal or unequal; cutting them in* L, K, O, M. *And likewiſe from any other point e, four more lines ol, ek, eo, em, be drawn to the ſides of the trapezium, reſpectively parallel to the former, cutting them in l, k, o, m. Then taking the rectangles of the oppoſite ſides, the rectangle* EL×EM : *el*×*em* : : EK× EO : *ek*×*eo.*

Thro' E, *e*, draw EQN, *eqn*, parallel to AB; and ERH, *erh*, parallel to AC. Then by ſimilar triangles.

ER : *er* : : EL : *el*, and EH : *eh* : : EM : *em*, and multiplying, ER×EH : *er*×*eh* : : EL×EM : *el*×*em*. Again

EQ : *eq* : : EK : *ek*, and EN : *en* : : EO : *eo*; and multiplying, EQ×EN : *eq*×*en* : : EK×EO : *ek*×*eo*. But (Prop. LXX.) ER×EH : *er*×*eh* : : EQ ×EN : *eq*×*en*. Therefore EK×EO : *ek*×*eo* : : ER× EH : *er*×*eh* : : EL×EM : *el*×*em*.

80. Cor. 1. *If* ABCD *be an inſcribed trapezium, and two oppoſite ſides* AB, CD *be produced to interſect in* X, *and the diagonals* AC, BD *be drawn to interſect in* Y; *and* X*e*YE *be drawn. Then* EX : *e*X : : EY : *e*Y.

For (in fig. 79.) ſuppoſe EL, EK, EO, EM, and *el*, *ek*, *eo*, *em*, all to lie in one line E*e*; and that AB, CD interſect in X, and AC, BD, in Y; and that E*e* paſſes thro' X, Y, (fig. 80.) which is a particular caſe of this Prop. then L, *l*, M, *m*, coincide in X; and K, *k*, O, *o*, meet in Y. Then the proportion $EL \times EM : el \times em :: EK \times EO : ek \times eo$, becomes $EX^2 : eX^2 :: EY^2 : eY^2$, whence $EX : eX :: EY : eY$. Fig. 80.

Cor. 2. *And if two tangents be drawn from* X, *and the line joining the points of contact paſſes thro'* **Y.** *It will be* $EX : eX :: EY : eY$. 81.

For let C approach to D, and A to B, till they coincide; then the lines AC, BD coincide in one, joining the points of contact, paſſing through Y; therefore it will ſtill be $EX : eX :: EY : eY$.

PROP. LXXII.

If BA *be the tranſverſe axis*; CD *the ſemiconjugate*; S, F, *the foci*, PM *an ordinate. Draw* DF, *and make* $DO = CP$, *and draw* OI *parallel to* DC; *then* $SM = BI$. 82.

For (Cor. 1. Prop. V.) $CA : CF :: CP : SM - CA$; therefore multiplying means and extremes, $CF \times CP = CA \times SM - CA^2$; and $CA \times SM = CA^2 + CF \times CP$. But by reaſon of the parallels, $DF : CF :: DO : CI$; that is, $CA : CF :: CP : CI$; whence multiplying, $CF \times CP = CA \times CI$; therefore $CA \times SM = CA^2 + CA \times CI$; and by equal diviſion, $SM = CA + CI = BC + CI = BI$.

Cor. 1. *The ſame conſtruction remaining*, $FM = AI$.

Cor. 2. *If the circle* CR *be deſcribed with the radius* SC, *to cut* SM *in* N, *and* NL *be drawn perp. to* BA. *Then* $BC - SL$, CD, *and* SM *are continually proportional. But if* PM *falls between* B *and* S, *it is* $BC + SL$.

For by ſimilar triangles $SM : SP :: SN$ or $SC : SL$; whence $SL \times SM = SC \times SP = CF \times SP = CF \times \overline{SC+CP} = CF^2 + CF \times CP$; but we had before $CF \times CP = CA \times SM - CA^2$; therefore $SL \times SM =$

Fig. $CF^2 + CA \times SM - CA^2$, and $CA \times SM - SL \times SM =$
82. $CA^2 - CF^2 = CD^2$; whence $CA - SL : CD :: CD : SM$.

Cor. 3. *Put z for the coſine of the angle* ASM, *then* $SM = \frac{CD^2}{CA - z \times CS}$. *And when* ASM *is greater than a right angle, then it is* $+ z \times CS$.

For it is (by trigonometry) Rad. (1) : SN or SC :: s.SNL (z) : $SL = z \times SC$.

PROP. LXXIII.

83. *As the tranſverſe axis, is to the conjugate :: ſo the area of a circle whoſe diameter is the tranſverſe, to the area of the ellipſis.*

Draw NMP parallel to the conjugate CD; then (by Prop. XIX.) the ordinate of the circle and ellipſis NP, MP; are every where to one another as CE or CA to CD. Therefore if AB be divided into an infinite number of equal parts, and ordinates erected, according to the method of indiviſibles. Then it will be as CE : CD :: ſo any ordinate PN : to the correſpondent ordinate PM :: (Prop. X.) ſum of all the ordinates PN : to the ſum of all the ordinates PM :: that is, as the area of the circle, to the area of the ellipſis.

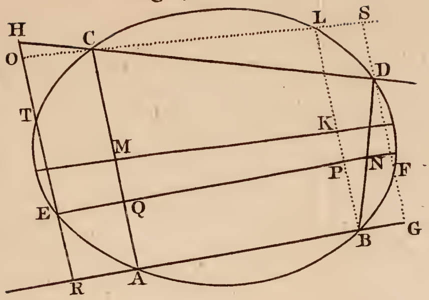
Cor. 1. *The area of a circle deſcribed on the conjugate : is to the ellipſis :: as the conjugate, to the tranſverſe.*

For the ſame reaſon as before. For all the correſponding ordinates are in that proportion, by Cor. Prop. XIX.

Cor. 2. *The correſpondent ſegments of the circle and ellipſis, are in the ſame proportion as the wholes.*

Cor.

Fig. 76.



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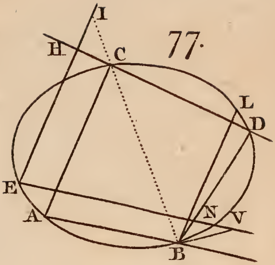

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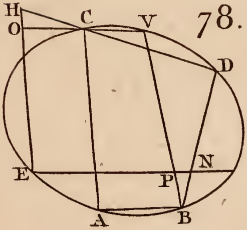

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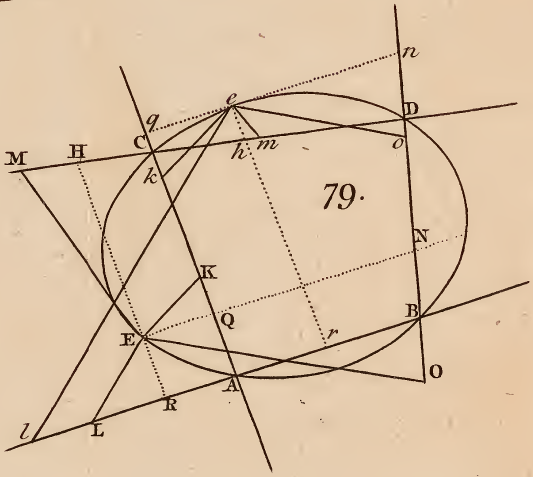

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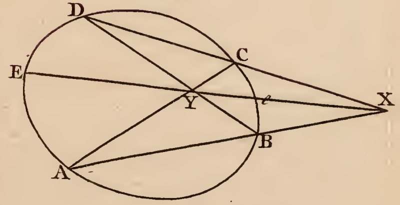

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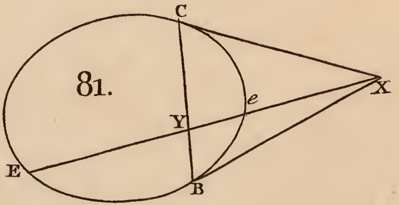


Pl. IX. *pa. 58.*

Cor. 3. *The ellipſis is a mean proportional, between two circles deſcribed on the tranſverſe and conjugate diameters.* Fig. 83.

For (Cor. 1.) the circle on the conjugate : ellipſis : : CD : CB : : (this Prop.) ellipſis : circle on the tranſverſe.

Cor. 4. *The ellipſis is equal to a circle whoſe diameter is a mean between the two axes.*

For circles being as the ſquares of the diameters; AB×DH will be the ſquare of the diameter of the mean circle, which is equal to the ellipſis. And hence,

Cor. 5. *An ellipſis : is to the rectangle of the two axes : : as any circle : to the ſquare of its diameter.*

Cor. 6. *The areas of ellipſes are to one another as the rectangles of their tranſverſe and conjugate axes.*

PROP. LXXIV.

The ſolidity of a ſpheroid, generated by an ellipſis revolving about either axis, is $\frac{2}{3}$ *the circumſcribing cylinder.* 84.

Let C be the center; CA $= t$, the axis of revolution; CD $= c$; AP $= x$, PM $= y$, $p =$ 3.1416. Suppoſe AC divided into an infinite number of equal parts, and planes be ſuppoſed to paſs thro' them, then according to the method of *Indiviſibles*, all theſe circular planes make up the ſolid. Now (by Prop. VI. and VII.), $tt : cc : : 2tx - xx :$ $yy = \frac{cc}{tt} \times \overline{2tx - xx} =$ PM^2, and $\frac{pcc}{tt} \times \overline{2tx - xx} =$ $p \times$PM^2, which is one of the circles conſtituting the ſolid. Now the ſum of all the $\frac{pcc}{tt} \times 2tx$ (Ar.

Inf.

Fig. 84. Inf. Prop. II.) is $\frac{pcc}{tt} \times \frac{2tx^2}{2}$. And the ſum of all the $\frac{pcc}{tt}xx$ (Inf. Prop. III.) is $\frac{pcc}{3tt}x^3$. Therefore the ſolidity of the ſegment AMF is $= \frac{pcc}{tt}x \times \overline{tx - \frac{1}{3}xx} = px \times \overline{\frac{1}{2}yy + \frac{ccxx}{6tt}}$. And when AP becomes AC, the ſolidity $= \frac{p \times CD^2}{CA^2} \times \frac{2}{3}AC^3 = \frac{2}{3}p \times CD^2 \times CA$. But the ſolidity of the cylinder, whoſe baſe is DE and height CA, is $= p \times CD^2 \times CA$, whence the ſpheroid is $\frac{2}{3}$ the cylinder.

Cor. 1. *The ſolidity of the ſegment* AMF $=$ *a cylinder whoſe height is* AP *or* x, *and baſe* $\frac{1}{2}$ *circle* MF $+ \frac{cc}{6tt} \times$ *circle* AP.

Cor. 2. *The ſolidity of the ſpheroid* DAE, *is to the ſphere whoſe radius is* AC; *as* DC^2 *to* AC^2; *and their correſpondent parts are in the ſame ratio.*

For the hemiſphere is $\frac{2}{3}p \times AC^3$ (Arith. Infin. Ex. 6.); and the ſegment, of the height AP, is $\overline{ptx - \frac{1}{3}pxx} \times x$, which is to the ſpheroidal ſegment, as tt to cc.

Cor. 3. *The ſolidity of the middle zone* MDEF *is* $=$ *to a cylinder whoſe baſe is* $\frac{2}{3}$ *circle* DE $+ \frac{1}{3}$ *circle* MF, *and height* CP.

Put $v = CP = \overline{t-x}$, then $x = t - v$, put this for x in the ſegment, and it becomes $\frac{pccxx}{tt} \times \overline{t - \frac{1}{3}x}$ $= \frac{pcc}{tt} \times \overline{t-v}^2 \times \overline{\frac{2}{3}t + \frac{1}{3}v} = \frac{2pcct}{3} - pccv + \frac{pccv^3}{3tt}$, ſubtract this from the whole DAE $= \frac{2}{3} pcct$, and Fig. there remains, $pccv - \frac{pccv^3}{3tt}$ for the zone DMFE. 84.

But $pccv - \frac{pccv^3}{3tt} = \frac{2}{3} pccv + \frac{1}{3} pccv - \frac{1}{3} \frac{pccv^3}{tt} = \frac{2}{3} pccv + \frac{1}{3} pv \times \overline{cc - \frac{ccvv}{tt}} = \frac{2}{3}$ the baſe DE $\times v + \frac{1}{3}$ the baſe MF $\times v$.

Cor. 4. *A ſpheroid whoſe axis is the tranſverſe : is to a ſpheroid whoſe axis is the conjugate : : as the coniugate : to the tranſverſe.*

For they are to one another as *cct* to *ttc*.

PROP. LXXV. *Prob.*

To deſcribe an ellipſis by motion.

1 *Way.*

Let CP, PD be two equal rulers moveable about 85. P; E a fixed point in PD. Then if CP be moved about the fixed point C, whilſt the end D moves along the right line AB. Then the point E will deſcribe an ellipſis AGEB.

It is plain by the conſtruction, that CG = CP —PE, and CB = CP+PE, make CO = CB, and draw EK parallel to AB. Then ſince CP = PD, therefore PK = PE = PO. And ſince the line PL biſſecting the angle KPE is perp. to KE and biſſects it; therefore OE which is parallel to it, will be perp. to KE or to AB. Therefore (Prop. XXXI.) the point E is in the ellipſis.

Cor. *The difference between the ſemitranſverſe and ſemiconjugate, or* CB—CG *is* = 2PE.

2 *Way.*

Fig.

2 *Way.*

23. Take the ruler MG, equal in length, to the ſemitranſverſe CA; and make MI = ſemiconjugate CD; then if the ruler be ſo moved that the end G may ſlide along CE, whilſt the point I ſlides along CA. The point M by its motion will deſcribe the ellipſis AMD.

This conſtruction is plain from Prop. XXX.

3 *Way.*

86. Take two rulers FE, SD, each equal to the tranſverſe AB, and a third ruler DE equal to FS the diſtance of the foci. And let DE be moveable about the points D, E. Then cauſe the rulers EF, DS, to move round, about the fixed points F, S. Then the interſection P will deſcribe an ellipſis APB.

Draw the tangent GPH, this (Prop. XIII. and Cor. 1.) biſſects the angle at P, and alſo DF, ES. Therefore the triangle FPG is ſimilar and equal to the triangle DPG; and SPH to EPH; therefore PE = PS, and FP + PS = FP + PE = FE = AB by conſtruction. Therefore P is in the curve of an ellipſis by Prop. I.

Cor. *If two lines* FE, SD, *equal to the tranſverſe* AB, *be drawn thro' any point* P *in the ellipſis, and the ends* D, E, *joined; then* DE = FS *the diſtance of the foci.*

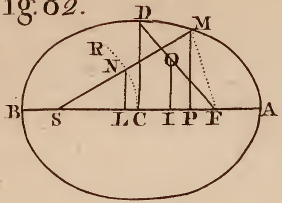
For the ſide DP = PF, and PE = PS, and $\angle$ P vertical; therefore DE = FS.

SCHOLIUM.

The ellipſis may alſo be deſcribed by definition 1.

PROP.

Fig. 82.



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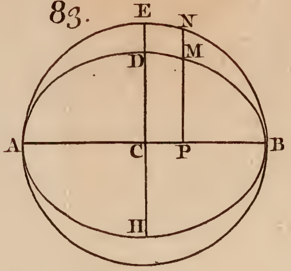

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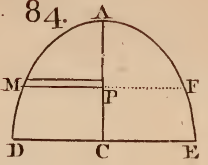

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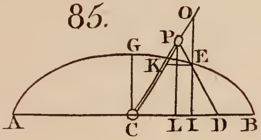

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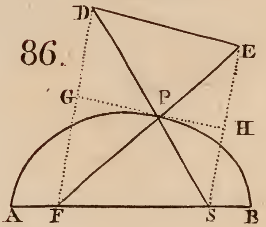

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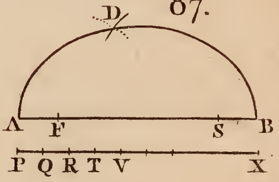

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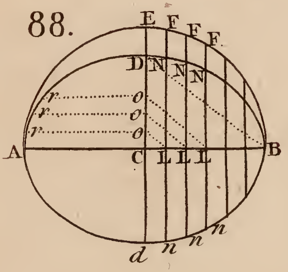

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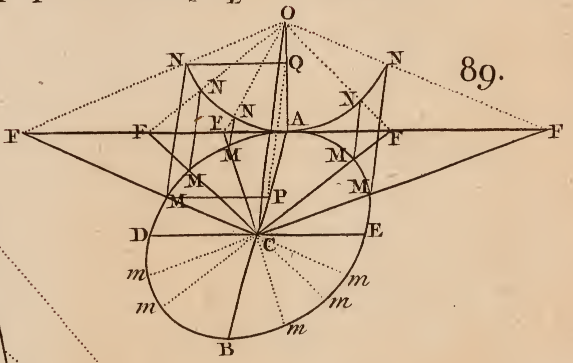

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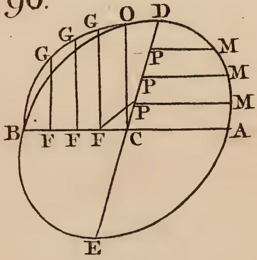

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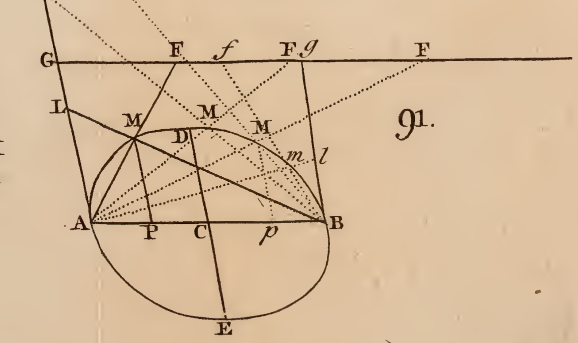


Pl X. *pa. 62.*

Fig.

PROP. LXXVI. *Prob.*

To deſcribe an ellipſis, by finding ſeveral points in the curve.

1 *Way.*

Let AB be the tranſverſe axis; F, S, the foci. 87.
Make PX = AB, and divide PX into as many equal parts as you pleaſe, as PQ, QR, RT, TV, &c. Then take any diſtance from P (greater than AF), as PV; and ſetting one foot in F, deſcribe the arch D. Then take the remainder XV, and ſet one foot in S, and croſs the former arch at D; then D is one point in the curve. Do ſo for all the other points R, T, &c. finding as many points D. Then a curve drawn thro' all the points D, making no angles, as ADB, will be the ellipſis required.

This conſtruction is plain from Prop. I.

2 *Way.*

Let AB be the tranſverſe or conjugate axis; di- 88.
vide AB into an infinite number of equal parts CL, LL, LL, &c. On the diameter AB, deſcribe the circle AEFB. On C the middle of AB, make CE perp. to AB. And thro' all the points L, draw the lines FL, FL, FL, &c. parallel to CE. Then in all the lines FL, make as the ſemitranſverſe CB, to the ſemiconjugate CD; ſo LF to LN; and produce NL to *n*, ſo that L*n* = LN. Then a regular curve drawn thro' all the points N, *n*, as ADNB*nd*A will be the ellipſis required.

This conſtruction is evident by Prop. XIX. and Cor.

3 *Way.*

About the axis AB, deſcribe the circle AEB, 88.
divide CB into any number of equal parts CL, LL,

Fig. 88. LL, LL, &c. and draw LF, LF, LF, &c. parallel to CE. Draw DB, and Lo, Lo, Lo, &c. parallel thereto. Or rather from L neareſt to C, draw Lo parallel to DB, and divide CD into parts, equal to Co, draw *or*, *or*, *or*, &c. parallel to AB, and each equal to its correſpondent LF; then *r*, *r*, *r*, &c. are in the curve of the ellipſis.

For by the property of the circle ALB $= LF^2$ $=$ (by conſtruction) or^2; therefore (Prop. XIV.) *r* is in the curve of the ellipſis.

PROP. LXXVII. *Prob.*

Given two conjugate diameters; to deſcribe the ellipſis by points.

1 *Method.*

89. Let AB, DE be the diameters; thro' A draw FF parallel to DE. And AO perpendicular to it, and equal to CD; and draw OC. About the center O with radius AO deſcribe the circle NAN. From C draw any number of right lines CF, CF, &c. to interſect FF; then from O to theſe interſections, draw OF, OF, &c. cutting the circle in N, N, &c. From N, N, &c. draw lines parallel to OC, to cut CM, CM, &c. in M, M, &c. Then M, M, M, &c. will be in the curve of the ellipſis.

And if MC, MC, &c. be produced till C*m* $=$ CM; then the points *m*, *m*, &c. will be in the ellipſis.

For draw MP, NQ parallel to FF, and join PQ; then by ſimilar triangles, CA : CP : : CF : CM : : OF : ON : : OA : OQ; therefore PQ is parallel to OC, and to MN; therefore NQ $=$ MP.

Then ſince CA : CP : : OA or CD : OQ; therefore $CA^2 : CP^2 :: CD^2$ or $ON^2 : OQ^2$; and by diviſion $CA^2 : CA^2 - CP^2 :: ON^2 : ON^2 - OQ^2$ or NQ^2;

Fig. 89. NQ^2; that is, $CA^2 : APB :: CD^2 : PM^2$. Therefore (Prop. XL.) the point M is in the ellipſis. And ſince $Cm = CM$; therefore (Prop. XXXII.) m is alſo in the curve.

2 *Method.*

90. Let AB, DE, be the two diameters. With radius CB and center C, deſcribe the circle BGO; let CO be perp. to CB; divide CB into any number of equal parts, at F, F, F, &c. and divide CD into the ſame number of equal parts at P, P, P, &c. at F, F, &c. draw FG, FG, &c. perpend. to BC. Thro' P, P, &c. draw PM, PM, &c. parallel to AB, each equal to the correſponding line FG. Then the points M will be in the ellipſis.

And if MP be produced beyond DC, ſo as to be equal to PM, theſe points will all be in the ellipſis BD.

For ſuppoſe BD drawn, and FP parallel to it. Then ſince CF is the ſame part of CB, as CP is of CD, then $CF : CP :: CB : CD$; whence $CB^2 : CD^2 :: CF^2 : CP^2$. And by diviſion $CB^2 : CD^2 :: CB^2 - CF^2 : CD^2 - CP^2$; but $BC^2 - CF^2 = BFA$ $=$ (nat. circle) $FG^2 =$ (conſtruction) PM^2; therefore $CB^2 : CD^2 :: PM^2 : CD^2 - CP^2$ or DPE or $CD^2 : CB^2 :: DPE : PM^2$. Whence (Prop. XL.) M is in the curve of the ellipſis.

3 *Method.*

91. Let AB, DE be the two diameters. Draw the tangent AL, find a third proportional to AB, DE, which ſet from A to G, this is the latus rectum of the diameter AB; draw GF parallel to AB. Thro' A, draw any number of right lines AF, AF, &c. Make AL, AL, &c. equal to the correſponding GF, GF; and from B, draw BL, BL, &c. to interſect AF, AF, &c. Then the interſections M, M,

Fig. M, &c. of the correſpondent lines AF, BL, will
91. be in the ellipſis.

And a like conſtruction may be made by means of the tangent Bg, at the end B. The reaſon being the ſame for both.

For draw the ordinate PM. Then by ſimilar triangles, GF : GA : : AP : PM. And BA : AL or GF : : BP : PM; and by multiplying, BA : GA : : APB : PM^2. Therefore (Prop. XL. Cor. 1.) the point M is in the ellipſis.

PROP. LXXVIII. *Prob.*

92. *A right line* DE *being given in an ellipſis; to find its diameter, and the center.*

Draw HI parallel to DE; biſſect DE in O, and HI in G. Thro' O, G, draw AOGB, which is the diameter. Biſſect AB in C, and C is the center.

For the diameter biſſects all the whole ordinates (Prop. XL. Cor. 3.); and the diameter is biſſected in the center (Prop. XXXII.)

PROP. LXXIX. *Prob.*

93. *Any diameter* AB *being given, in an ellipſis; to find its conjugate.*

Draw HI parallel to AB. Biſſect AB, HI, in C and G. Thro' C and G draw the conjugate DE.

PROP. LXXX. *Prob.*

93. *Any diameter* AB *being given; to draw an ordinate to it from a given point in the curve as* H.

Thro' the center C, draw HK; and KL parallel to AB; then draw LH which will be the double ordinate required.

For ſince HC = CK, and AC, LK parallel; therefore HO = OL. Fig. 93.

Or thus,

Draw HI parallel to AB. Draw DE through the middle of AB, HI; and draw HL parallel to it; then HO is the ordinate ſought.

For DE is the conjugate, by the laſt Prop. and HO is parallel to it.

PROP. LXXXI. *Prob.*

An Ellipſis being given to find the two axes. 94.

Draw two parallel lines thro' the ellipſis as FK, ML; biſſect them, and thro' the middle points draw IN, biſſect IN in C, for the center. On the center C deſcribe an arch of a circle, to cut the ellipſis in two points G, and H. Draw GH and biſſect it; then thro' C and the middle point of GH, draw BA for the tranſverſe axis; alſo through C draw DE parallel to GH, for the conjugate.

For AB paſſing thro' C muſt be a diameter; and ſince CG is = CH, CB paſſing thro' the middle of GH, will be perp. to it, and alſo perp. to the diameter DE (which is parallel to FG); therefore AB, DE are the axes.

PROP. LXXXII. *Prob.*

An ellipſis being given; to find two diameters, making a given angle with one another, if poſſible. 95.

Draw any diameter FG, upon FG deſcribe the ſegment of a circle FNG, to contain the angle given (or its ſupplement, for it will be all one), to cut the ellipſis in N; draw NG, NF and biſſect them, and thro' the points of biſſection draw the

Fig. diameters AB, DE; theſe will be the diameter
95. required. If the circle does not cut the ellipſis the problem is impoſſible.

For ſince GN, FN are biſſected, they will b ordinates to CD, CB. And ſince FG is alſo biſſected in C, therefore CD is parallel to FN, an CB to GN; whence < BCD = FNG = the give angle.

PROP. LXXXIII. *Prob.*

From any given point M *in an ellipſis, to dra a tangent.*

1 *Way.*

96. From the given point M to the center dra MC, and find its conjugate CB (by Prop. LXXIX.) then from M draw MG parallel to CB, for th tangent.

2 *Way.*

Draw the ordinate MP to the diameter CF; the take CP : CF : CG continual proportionals; and draw MG for the tangent, by Prop. XLVI.

3 *Way.*

97. Draw two lines MF, MS, from the given point to the foci; make MD = ME of any length, an draw DE, then from the point M draw MT parallel to ED, for the tangent at M.

For ſince EM = MD, the < D = < E. And ſince MT is parallel to ED, the < TMD = MDE = MED = EMR; therefore (Prop. X.) MT is a tangent at M.

4 *Way.*

96. If KF be one of the axes, deſcribe a circle abou KF as a diameter, and where it cuts the ordinat PM (produced), as at m, draw the tangent mG t the circle; then draw MG which will touch th ellipſis at M, by Prop. XVIII.

5 *Way.*

5 *Way.*

Let F, S, be the foci, produce FM to G, ſo 97.
that MG = MS, draw GS and biſſect it in O. Then MO drawn, will touch it in M, by Prop. XIII.

PROP. LXXXIV. *Prob.*

From a point given without the ellipſis, to draw a tangent to it.

1 *Way.*

Let G be the point given, thro' G and C draw 96.
the diameter FK. Make CG, CF, CP continual proportionals, and at P draw the ordinate PM to the diameter FK; then GM will be a tangent. By Prop. XLVI.

2 *Way.*

If KF be the axis, draw CG, on which deſcribe the ſemicircle C*m*G, and on the diameter KF deſcribe an arch F*m* to cut the former in *m*. Draw *m*P perp. to CG, to cut the ellipſis in M. Then draw GM for the tangent. By Prop. XVIII.

3 *Way.*

Let T be the point given; thro' T draw the 98.
iameter AB, and (Prop. LXXIX.) find its conugate CD. Draw any line TS = TB, and make TR = TC. Draw RA, and SP parallel to it, cutting the diameter AB in P. Draw the ordinate PM parallel to CD; then TM is the tangent equired.

For by ſimilar triangles, TS : TR : : TP : TA; hat is by conſtruction, TB : TC : : TP : TA; and by diviſion, BC : TC : : AP : AT; and by diviion again BC : TC : : BC—AP or CP : TC—AT or CA; that is, CP : CA : : CA : CT; therefore Prop. XLVI.) TM is a tangent.

Fig.

PROP. LXXXV. *Prob.*

99. *There is given the focus* F; *and alſo three points o*
the ellipſis, M, N, P; *to deſcribe the ellipſis.*

Draw NMK thro' two of the points, and NP
alſo thro' two; and draw MF, NF, PF, to th
focus. Then make FN : FM : : NK : MK. Alſ
FN : FP : : NI : PI. Then draw KI, and M
perp. to it. And FT perp. to it from the focus
and make FA : AT : : FM : MH. Then produc
TF to B, ſo that BF may be to BT in the ſam
ratio. And AB will be the axis, and the middl
point C, the center. And make CS = CF, an
S will be the other focus. Whence the ellipſi
may be deſcribed by the foregoing propoſitions.

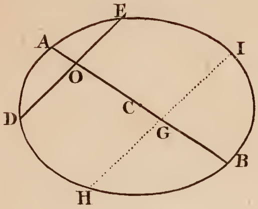
Draw NG, PL, perp. to KT. Then by con
ſtruction FN : FM : : NK : MK : : (ſimilar tr
angles) NG : MH. And FN : FP : : NI : PI :
NG : PL. Alſo FA : AT : : FM : MH. Ther
fore FN : NG : : FP : PL : : FM : MH : : FA
AT : : FB : BT. Which (Prop. XXIX. and Schol.
is the property of the ellipſis, where KT is the d
rectrix; and therefore the points M, N, P, a
in the ellipſis. Alſo ſince BT : BF : : AT : A
Therefore AT : AF : : BT—AT : BF—AF :
BA : SF : : CA : CF; which is the property
the directrix.

PROP. LXXXVI. *Prob.*

100. *Given two conjugate diameters of an ellipſis;*
find the axes.

Let the given diameters be HI, LM; produ
CH, and make CH, CM, HN continual propo
tionals; let GK be a tangent at H; biſſect CN
H

Fig 92.



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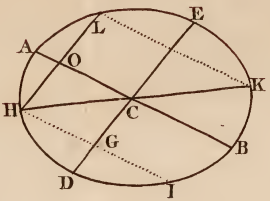

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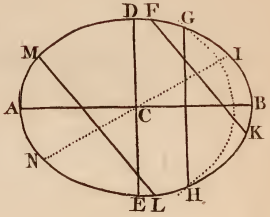

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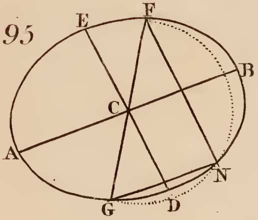

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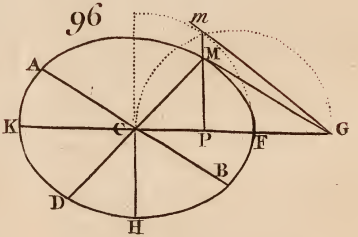

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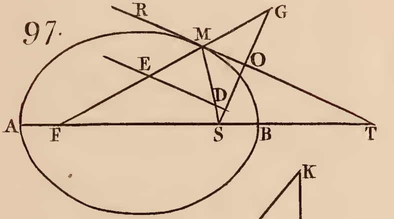

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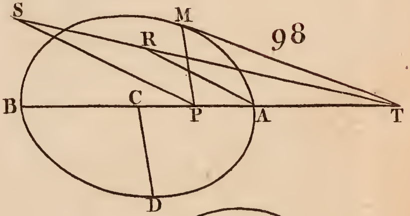

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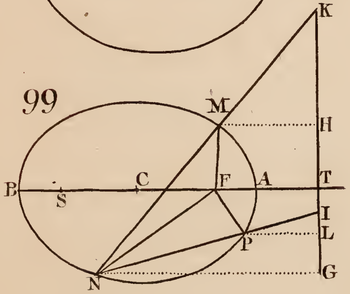

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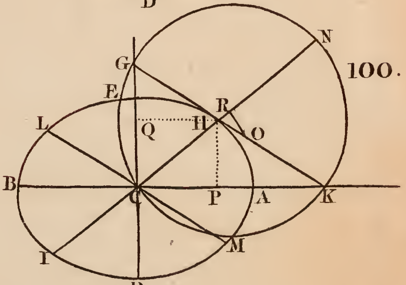


Pl.XI. *pa. 70.*

R, and draw RO perp. to it interſecting GK in O. Fig.
About the center O, with radius OC deſcribe the 100.
circle CGNK, cutting GK in G, and K. Thro' K, C, and G, C, draw AB, ED, for the two axes.

For by conſtruction $CL^2 = CHN =$ (Geom. IV. 20. Cor. 2.) GHK; therefore (Prop. LIII.) AC, EC are conjugates. And ſince GOK is the diameter of the circle; GCK is a right angle, and therefore AC, EC are the ſemiaxes.

Cor. *If* HP, HQ *be drawn parallel to* EC, AC. *And there be taken* CA *a mean between* CP *and* CK; *and* CE *a mean between* CQ *and* CG, *you'll have the length of the axes.*

By Prop. XVI, XVII. Becauſe GHK is a tangent at H.

PROP. LXXXVII. *Prob.*

To deſcribe an ellipſis about a given parallelogram 101.
EGHF, *to paſs thro' a given point* P.

Biſſect the oppoſite ſides of the parallelogram EF, GH; and EG, FH; by the conjugate diameters AB, ID. Draw PQ parallel to ID; and take $AC^2 = \dfrac{CL^2 \times PQ^2 - CQ^2 \times EL^2}{PQ^2 - EL^2}$, and $CD^2 = \dfrac{CL^2 \times PQ^2 - CQ^2 \times EL^2}{CL^2 - CQ^2}$. And AC, CD being had, AL, and its equal KB will be had, from whence the ellipſis is eaſily deſcribed.

For (Prop. XL.) $AC^2 : CD^2 :: ALB : EL^2 :: AQB : PQ^2$. That is, $AC^2 : CD^2 :: AC^2 - CL^2 : EL^2 :: AC^2 - CQ^2 : PQ^2$. Hence $\overline{AC^2 - CL^2} \times PQ^2 = \overline{AC^2 - CQ^2} \times EL^2$; therefore $AC^2 \times \overline{PQ^2 - EL^2} = CL^2 \times PQ^2 - CQ^2 \times EL^2$. Whence AC is determined.

Fig. 101. Again, ſince $AC^2 : CD^2 :: AC^2 - CL^2 : EL^2$; by diviſion, $AC^2 : CD^2 :: CL^2 : CD^2 - EL^2$. And ſince $AC^2 - CL^2 : EL^2 :: AC^2 - CQ^2 : PQ^2$; by diviſion, $AC^2 - CL^2 : EL^2 :: CQ^2 - CL^2 : EL^2 - PQ^2$. Therefore $CL^2 : CD^2 - EL^2 :: CQ^2 - CL^2 : EL^2 - PQ^2$. And by addition, $CL^2 : CD^2 - EL^2 :: CQ^2 : CD^2 - PQ^2$; therefore $CL^2 \times \overline{CD^2 - PQ^2} = CQ^2 \times \overline{CD^2 - EL^2}$; whence $CD^2 \times \overline{CL^2 - CQ^2} = CL^2 \times PQ^2 - CQ^2 \times EL^2$. And therefore CD is determined.

Cor. *Hence* $AC^2 : CD^2 :: CL^2 - CQ^2 : PQ^2 - EL^2$. For $AC^2 \times \overline{PQ^2 - EL^2} = CL^2 \times PQ^2 - CQ^2 \times EL^2 = CD^2 \times \overline{CL^2 - CQ^2}$.

PROP. LXXXVIII.

102. *If a cylinder be cut obliquely by a plane; the ſectio will be an ellipſis.*

Let AGH, RIS, be the baſes of the cylinder AMB the ſection; AB its diameter interſecting th axis of the cylinder KO in C; then ſince AO = OH, therefore AC = CB. Take any point P, anc ſuppoſe two planes parallel to the baſe AGH, t paſs thro' C and P, cutting the ſection in the line CD, PM. Thro' CD, PM, draw the planes KG NE. If the plane RH be ſuppoſed perp. to AGI and AMB, then (Geom. V. 15.) CD, PM will b perp. to AB, and GO, EF, to AH. And KI = CD = OG, and NL = PM = FE. By the na ture of the circle, the rectangle AFH = FE^2 = PM^2. Then by ſimilar triangles, $AC : AO :: AP$ AF, and $AC : AO :: PB : FH$; then multiplyin $AC^2 : AO^2$ or $CD^2 :: APB : AFH$ or PM^2. Ther fore (Prop. VI. Cor. 1.) the curve AMB is an e lipſis.

Co

Cor. *If the plane be parallel to the baſe of the cylinder, the ſection will be a circle.* Fig. 102.

PROP. LXXXIX.

If an oblique cone ABD *be cut by a plane* EF, *in ſub-contrary poſition, the ſection* ELF *will be a circle.* 103.

Let the plane ABD be drawn thro' the axis of the cone AC, perp. to the plane of the baſe BGD. And let another plane ELF alſo perp. to ABD, cut the cone, ſo that the angle AFE may be equal to ABD. Then the cone is ſaid to be cut in *ſub-contrary poſition.*

Thro' any point R in the diameter EF, draw the plane HLK, parallel to the baſe BGD, cutting the plane ELF, in the line LR. Alſo let another plane ACG, cut the plane HCK, in the line OM. Then (Geom. V. 11.) OM and CG are parallel; whence by ſimilar triangles, CD : OK :: AC : AO :: CG or CD : OM; whence OK = OM; therefore EMK is a circle.

Again (Geom. V. 15 and Cor.) LR is perp. to the plane ABD, and alſo to EF and HK; and ſince the angle KFR = EBD or EHR; therefore the triangles ERH, KRF are ſimilar, and ER : HR :: KR : FR. And ERF = HRK = (Geom. IV. 17.) RL^2; therefore ELF is alſo a circle.

Cor. *If a cone be cut by a plane parallel to the baſe, the ſection* (HLK) *will be a circle.*

This appears from the demonſtration.

PROP. XC.

If a cone ABC, *be cut by a plane paſſing thro' both its ſides; the ſection* BNDK *will be an ellipſis.* 104.

Draw the plane ABC thro' the axis of the cone, and let the plane DNB paſs thro' the tangent at B, Fig. 104. and let $BH = HD$, and thro' H and any point I, of the diameter BD, draw two planes, parallel to the baſe of the cone BC; cutting the plane BNDK in the lines KN, LQ, which will be parallel to one another (by Geom. V. 11.); and the ſections FNGK, and OQPL will be two circles (by Cor. to the laſt Prop.).

Then by the nature of the circle, $OI \times IP = IQ^2$ or IL^2, and $FH \times HG = KH^2$ or HN^2. By the ſimilar triangles DIP, DHG, and BIO, BHF; $DI : IP :: DH : HG$, and $BI : IO :: BH : HF$. And by multiplying, $DI \times IB : OI \times IP$ or $IQ^2 :: BH \times HD : FH \times HG$ or HN^2. Therefore (Prop. XL.) the curve BNDK is an ellipſis.

Cor. 1. *The ſquare of the conjugate is equal to the rectangle of the diameters of the fruſtum*: $KN^2 = ED \times BC$.

For draw DE parallel to BC; then by ſimilar triangles, ſince DB is biſſected in H, therefore $FH = \frac{1}{2} ED$, and $GH = \frac{1}{2} BC$; and the rectangle $FHG = KH^2 = \frac{1}{4} BC \times ED$; and $KN^2 = BC \times ED$.

Cor. 2. *The latus rectum of the ſection is* $= \frac{ED \times BC}{BD}$.

For (Prop. IV. Schol.) the latus rectum is equal to $\frac{KN^2}{BD}$.

105. Cor. 3. *If it be a right cone, the ſquare of the tranſverſe is equal to the ſquare of the ſide of the fruſtum, and the rectangle of the diameters of the baſes.* $BD^2 = DC^2 + ED \times BC$.

Let DW be perp. to BC; then $WC = \frac{BC - ED}{2}$.

And (Geom. II. 23.) $BD^2 = BC^2 + CD^2 - 2BCW = BC^2 + CD^2 - 2BC \times \frac{BC - ED}{2} = CD^2 + BC \times ED$.

Cor. 4. *In a right cone, the diſtance of the focus* S, *from the center, is equal to half the ſide of the fruſtum* : HS = FB. Fig. 105.

For (Cor. 3.) $BD^2-CD^2 = BC\times ED =$ (Cor. 1.) KN^2. And dividing by 4, $BH^2-BF^2 = HN^2$, and $BF^2 = BH^2-HN^2 =$ (Prop. II.) HS^2, and BF = HS.

Cor. 5. *If* AI *be perp. to the baſe of the cone, and* AB *perp. to* MH *the diameter of the ellipſis. And if* AN = AB, *and* FND *be drawn parallel to the baſe* CM. *Then* FD *is = the latus rectum; and* DS *(parallel to* MH*) is =* HZ *the diameter at top.* 106.

Draw AL, AG parallel to MC, MH, and let R = latus rectum; then the triangles ABL, ANG are ſimilar and equal, and AG = AL. The triangles MCH, DFS, GFA, are ſimilar; whence MC : MH : : DF : DS : : GF : GA or AL : : (ſimilar triangles GFA, LAH) AF : AH : : (ſimilar triangles AFD, AHZ) FD : HZ; therefore DS = HZ, and $MH\times FD = MC\times HZ =$ (Cor. 1.) ſquare of the conjugate = (IV. Schol.) $MH\times R$, therefore FD = R.

Cor. 6. *In a right cone, if* F *be the focus,* C *the center of the ellipſis;* AG = AF. *The circle* GOI *parallel to baſe of the cone, will cut the plane of the ellipſis* CAH *in the line* ED, *the directrix of the ellipſis.* 107.

For (Prop. XXVIII. Cor. 2. and XXIX. Schol.) if DH be the focal tangent, DE will be the directrix; where CH = CA, and AF, are ſet perp. to AM. But if they be not perp. but in any other ſituation parallel to each other, as CB and AG; then BG will cut AM in the ſame point D, and ſtill DE is the directrix. And DB will be parallel to MN; for (Cor. 4.) CF = AP or $\frac{1}{2}$AN; and PC (drawn thro' the middle of AN, AM) will be parallel MN. And (XVI.) CA : CD : : CF : CA : :

Fig. CA : : (by diviſion) AF : AD; that is, PA : CA : :
107. AG : AD; therefore DG is parallel to PC or MN.

108. Cor. 7. *In a right cone, if* AG = AF, *and the circle* GNI *be drawn parallel to the baſe. Then the diſtance of any point* P *from the focus,* FP; *will be equal to* PN *the diſtance of* P *from the periphery of the circle* GNI.

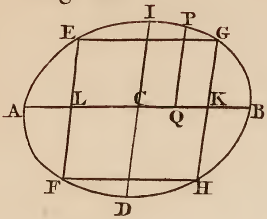
For DE is the directrix; draw DR the focal tangent; then (Prop. XXVIII.) if the ordinate PQ be produced to the tangent at R, then QR = FP; but QL parallel to AG, being made equal to QR (as in the laſt Cor.), then DL will be parallel to the baſe HM. Whence FP = QL = PB (becauſe the plane SPQR is parallel to the plane DBL) = SG = PN.

Cor. 8. *And ſince there are two foci; another ſuch circle drawn the ſame way below the baſe* HM, *will have the ſame property. And their diſtance along the ſide of the cone will be equal to the tranſverſe axis.*

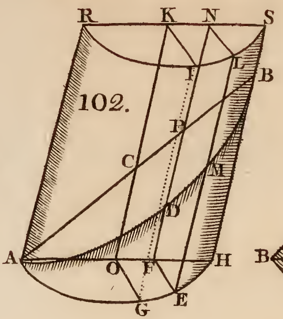
SCHOLIUM.

Hence appears the reaſon why the *Ellipſis* is called one of the *Conic Sections.* For a cone cut by any plane which paſſes thro' both its ſides, will always make the figure of an ellipſis, whoſe properties have been demonſtrated in this book; except it happen in ſome particular caſes, that the two axes of the ellipſis be equal; and then the foci coincide in the center; and the ellipſis becomes a circle; which therefore may be reckoned one ſort of an ellipſis.

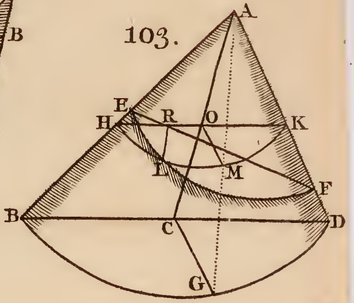
Fig. 101.



102.



103.



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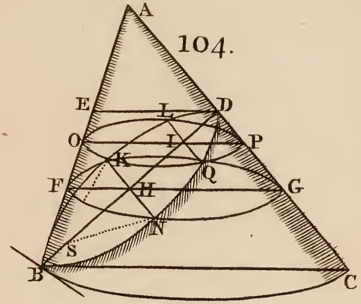

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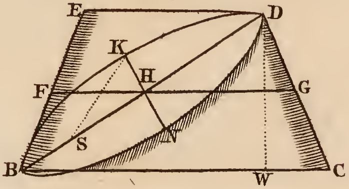

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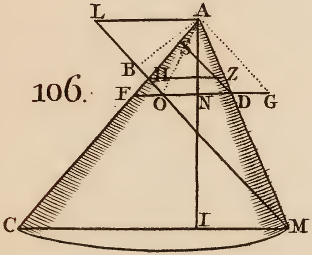

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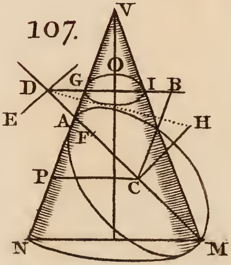

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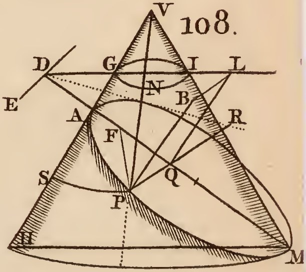


Pl. XII. *pa. 76.*

CONIC SECTIONS.

BOOK II.

Of the HYPERBOLA.

DEFINITIONS.

DEFIN. I.

IF the ends of two threads SPQ, FPQ, be faſtened at the points S, F. And be made to paſs thro' a ſmall bead, or pin P, and knotted together at Q. Then taking hold of Q, and drawing the threads tight; if the bead be moved along the threads; the point P will deſcribe the curve *mp*APM, called an *Hyperbola*. Fig. 1.

DEF. II.

And if the end of the long thread be fixed at F, and that of the ſhort one at S; and the curve NBR be deſcribed after the ſame manner; that curve is called the *oppoſite Hyperbola*; and both curves together, MA*m*, NBR are called *oppoſite Sections*, or *oppoſite Hyperbolas*.

DEF. III.

The two fixed points F, S, are called the *Foci*.

DEF. IV.

The line AB (paſſing thro' the foci, when continued), contained between the two parts of the curve, is called the *tranſverſe Axis*.

DEF.

Fig. 1.

DEF. V.

The middle point of AB, that is C, is called the *Center* of the hyperbola, or of the oppoſite ſections.

DEF. VI.

If VY be drawn thro' the center C perp. to AB; and with radius CF, and center A, an arch be deſcribed, cutting VY in V and Y; then VY is called the *conjugate Axis.*

DEF. VII.

2. Any line TO drawn thro' the center C, and terminated at the oppoſite ſections, is called a *Diameter*; and the extremity T (or O) its *Vertex.* And the line drawn thro' the center, parallel to the tangent at the vertex, is called its *conjugate Diameter.*

DEF. VIII.

If any diameter OT is continued within the curve, the part within, TP, is called the *Abſciſſa.*

DEF. IX.

Any line PM, drawn parallel to the tangent at the vertex T, and terminated at the abſciſſa and curve; is called an *Ordinate* to that diameter TO. And if it go quite thro' the curve, it is called a *double Ordinate.*

DEF. X.

1. The line LI, drawn thro' the focus F, perp. to the tranſverſe axis AB, and terminating at the curve, is called the *Parameter* or *Latus Rectum.*

DEF. XI.

If the ends of the two axes be joined by the lines BY, BV; and thro' the center C, two lines

CH, CG, be drawn parallel to BY, BV; or (which is the ſame) if VY be placed at A perp. to BA; and the lines CH, CG, be drawn from the center C, thro' the ends E, D; theſe lines CH, CG, are called the *Aſſymptotes*, of the hyperbola, or of the oppoſite hyperbolas. Fig. 1.

DEF. XII.

When the tranſverſe and conjugate axes are equal, AC = CV or AD, the curve is called an *equilateral Hyperbola*, or *right angled Hyperbola.*

DEF. XIII.

A right line which meets the hyperbola in one point T, but does not cut it, as TH, is called a *Tangent* to it, in that point T. 2.

DEF. XIV.

If two oppoſite hyperbola's KO, TW, be in like manner deſcribed to the tranſverſe VY (= DE), and conjugate AB; theſe are called *conjugate Hyperbolas*, with regard to the former.

PROP. I.

The difference of the lines SP, FP, *drawn from the foci, to any point* P *of the curve, is equal to the tranſverſe axis* AB. 1.

For by conſtruction PS — PF = AS — AF = AB + BS — AF = (becauſe BS = AF) AB.

Cor. *Hence* CF = CS, *or the foci are equally diſtant from the center.*

PROP.

Fig.

PROP. II.

1. *The ſquare of the diſtance of the focus from the center, is equal to the ſum of the ſquares of the ſemi-tranſverſe and ſemiconjugate.* $CF^2 = CA^2+CY^2$.

For make AE equal and parallel to CY, then the radius $CE = CF$; and in the right angled triangle CAE, $CE^2 = CA^2+AE^2$; that is, $CF^2 = CA^2+AE^2 = CA^2+CY^2$.

Cor. $CF^2 - AE^2 = CA^2$; *and* $CF^2 - CA^2 = AE^2 = CY^2$.

PROP. III.

1. *The rectangle of the focal diſtances from either vertex, is equal to the ſquare of the ſemiconjugate,* $FA \times SA = CY^2$.

For making $AE = CY$; by the property of the circle, $FA \times AS = AE^2 = CY^2$.

Cor. *The rectangle of the diſtance of either focus from the two vertices, is equal to the ſquare of the ſemiconjugate,* $FA \times FB = AE^2 = CY^2$.
For $SB = FA$, and $SA = FB$; whence $FA \times FB = FA \times SA = AE^2$.

PROP. IV.

3. *As the tranſverſe axis, is to the conjugate; ſo the conjugate, to the latus rectum of the tranſverſe;* AB : VY : : VY : LI.

For (Prop. I.) $SL - LF = BA = 2CA$; and $SL = 2CA+LF$; and $SL^2 = 4CA^2+4CA \times LF+LF^2$: and in the right angled triangle SLF,

$SL^2 = SF^2 + LF^2$, and ſubtracting LF^2 from these two values of SL^2; then $4CA^2 + 4CA \times LF = SF^2 = 4CF^2$; and $CF^2 = CA^2 + CA \times LF$. But (Prop. II.) $CF^2 = CA^2 + CY^2 = CA^2 + CA \times LF$; therefore $CY^2 = CA \times LF$; and multiplying by 4, $VY^2 = BA \times LI$. Fig. 3.

Cor. 1. *As the ſemitranſverſe, to the ſemiconjugate, ſo the ſemiconjugate to half the latus rectum*, $CA : CY : LF$.

Cor. 2. *As the ſemitranſverſe, to the diſtance of the focus from the center; ſo is the ſame diſtance, to the ſum of the ſemitranſverſe and half the latus rectum*, $CA : CF :: CF : CA + LF$.

For (Prop. II.) $CF^2 = CA^2 + CY^2 = CA^2 + CA \times LF = CA \times \overline{CA} \times \overline{FL}$.

Cor. 3. *The rectangle* BFA $= \frac{1}{2}$ *tranſverſe* $\times \frac{1}{2}$ *latus rectum* $= CA \times FL$.

By Cor. 1. and Prop. III.

SCHOLIUM.

Since the tranſverſe axis is to the conjugate, as the conjugate to the latus rectum of the tranſverſe axis. Therefore in any other diameters, the third proportional to any diameter and its conjugate, is called the *Latus Rectum of that Diameter*. Therefore in a right angled hyperbola, every diameter is equal to its latus rectum.

PROP.

Fig.

PROP. V.

4. *From any point* M *in the curve, draw the lines,* MF, MS, *to the foci; and the ordinate* MP *perp. to the tranſverſe axis* AB. *Then,*
As the ſemitranſverſe, CA :
to the diſtance of the focus from the center, CF : :
So the diſtance of the ordinate from the center, CP :
to half the ſum of the lines MS, MF, *or* $\frac{MS+MF}{2}$.

Make $SD = CA$; then $SM = CA+DM$, and $FM = SM-2CA = DM-CA$. In the right angled triangle SMP, $SM^2 = CA^2+2CA\times DM+DM^2 = SP^2+PM^2 = \overline{CF+CP}^2 + PM^2 = CF^2+2CF\times CP+CP^2+PM^2$.

And in the right angled triangle FMP, $FM^2 = DM^2-2DM\times CA+CA^2 = FP^2+PM^2 = \overline{CP-CF}^2+PM^2 = CP^2-2CF\times CP+CF^2+PM^2$.

And ſubtracting the latter equation from the former, $SM^2-FM^2 = 4CA\times DM = 4CF\times CP$. And $CF\times CP = CA\times DM$.

But ſince $SM = DM+CA$, and $FM = DM-CA$; therefore $SM+FM = 2DM$, and $DM = \frac{SM+FM}{2}$; whence $CF\times CP = CA\times\frac{SM+FM}{2}$.

Cor. 1. *If* F, S, *be the foci,* MP *an ordinate; then it is* CA : CF : : CP : SM—CA *or* FM+CA.

For $CF\times CP = CA\times DM$, and $DM = SM-CA = FM+CA$.

Cor. 2. *If* F, S, *be the foci,* MP *an ordinate, then the difference of the ſquares of the lines* SM, FM; *that is,* $SM^2-FM^2 = 4CF\times CP$.

Cor. 3. *If* F, S, *be the foci*, MP *an ordinate*; *then* $CA \times \overline{SM+FM} = 2CF \times CP$. Fig. 4.

For $SM^2 - FM^2 = \overline{SM+MF} \times \overline{SM-MF} = 2CA \times \overline{SM+MF} = 4CF \times CP$. And $CA \times \overline{SM+FM} = 2CF \times CP$.

SCHOLIUM.

When P falls between A and F, then $FP = CF - CP$, and its ſquare the ſame as before, and the reſt of the demonſtration the ſame.

PROP. VI.

If MP *be an ordinate on the tranſverſe axis, it will be*, 4.
As the ſquare of the tranſverſe, BA^2 :
to ſquare of the conjugate, NE^2 : :
So rectangle of the ſegments of the tranſverſe, BPA :
to ſquare of the ordinate, PM^2.

Make $SD = CA$, then $DM = \frac{SM+FM}{2}$. And (Prop. V.) CA : CF : : CP : DM. And by compoſition, CA : CA+CF or BF : : CP : CP+DM; and alternately, CA : CP : : BF : CP+DM. And again by compoſition, CA : CA+CP or BP : : BF : BF+CP+DM.

But $BF = BC+CF = SD+CF$. And $BF+CP+DM = SD+CF+CP+DM = SM+SP$. Therefore CA : BP : : BF : SM+SP.

Again, ſince CA : CF : : CP : DM; by diviſion CA : CF−CA or AF : : CP : DM−CP; and alternately, CA : CP : : AF : DM−CP; and by diviſion, CA : CP−CA or AP : : AF : DM−CP−AF.

Fig. 4. But $AF = CF - CA = SC - SD$; therefore $DM - CP - AF = DM - CP + SD - SC = SM - SP$; therefore,

$$CA : AP :: AF : SM - SP$$

and before $CA : BP :: BF : SM + SP$,

and multiplying, $CA^2 : AP \times BP :: AF \times BF : SM^2 - SP^2$.

Make CN or CE perp. to AB, and equal to half the conjugate; then (Prop. III.) $AF \times BF = CN^2$. And $SM^2 - SP^2 = PM^2$. Whence $CA^2 : AP \times PB :: CN^2 : PM^2$ or $BA^2 : NE^2 :: BPA : PM^2$.

Cor. 1. *As ſquare of the ſemiconjugate : to the ſquare of the ſemitranſverſe : : rectangle of the diſtances of the ordinate from the vertexes : to ſquare of the ordinate.* $CA^2 : CN^2 :: BPA : PM^2$.

Cor. 2. *As the tranſverſe axis, to the latus rectum : : ſo the rectangle of the ordinates diſtances from the vertexes, to the ſquare of the ordinate.* BA : *lat. rect.* : : BPA : PM^2.

For BA : lat. rect. : : BA^2 : BA × lat. rect. : : (Prop. IV.) $BA^2 : NE^2$: : (this Prop.) $BPA : PM^2$.

Cor. 3. *The rectangles of the (abſciſſa's or) diſtances of the ordinates from the vertices of the figure, are as the ſquares of theſe ordinates.*

For each rectangle is to the ſquare of its ordinate, in a given ratio.

Cor. 4. *As the ſquare of the ſemitranſverſe : to the rectangle of the focal diſtances from the vertex : : ſo rectangle of the ordinate's diſtances from the vertexes : to the ſquare of the ordinate.*

Cor. 5. *The whole ordinates to the tranſverſe, are biſſected by the axis. And equal ordinates are equally diſtant from the center.*

For the rectangle of the abſciſſas on the tranſverſe axis being in a given ratio to the ſquare of the ordinate, and that rectangle being the ſame for the ordinates on both ſides; theſe ordinates will therefore be equal. Again, that rectangle remains the ſame, at equal diſtances on each ſide the center; and conſequently the ordinates are equal at equal diſtances. Fig. 4.

PROP. VII.

If MP *be an ordinate to the tranſverſe axis*; C *the center*, F, S *the foci*; *and making* CA : CF : : CP : CI; *then the rectangle* $SM \times MF = CI^2 - CA^2$. 4.

Make DS = CA; then (Prop. V.) $CF \times CP = CA \times \frac{MS+MF}{2} = CA \times CI$, by conſtruction. But $SM = DM + CA$, and $FM = (SM - 2CA)\ DM - CA$; therefore $\frac{MS+MF}{2} = DM$, and $CA \times DM = CA \times CI$; whence $CI = DM$. But $SM \times FM = \overline{DM+CA} \times \overline{DM-CA} = DM^2 - CA^2 = CI^2 - CA^2$.

Cor. 1. *If* CA : CF : : CP : CI; *then* SM = BI, *and* FM = AI.

For DM = CI, and $SM = CA + CI = BI$, and $FM = CI - CA = AI$.

Cor. 2. *The rectangle* $SM \times FM = BI \times AI$. From Cor. 1.

Cor. 3. CA : CF : : CP : CA + FM : : CP : SM − CA.

Fig. 4.

PROP. VIII.

If F, S *be the foci*; C *the center*; MP *an ordinate to the tranſverſe axis*; CE *or* CN *the ſemiconjugate. Then* $CM^2 = CA^2 - CE^2 + SMF$.

For in the triangle SMF (Geom. II. 28.) $SM^2 + MF^2 = 2CF^2 + 2CM^2$. But $\overline{SM - MF}^2 = SM^2 - 2SM \times MF + MF^2$, and $SM^2 + MF^2 = \overline{SM - MF}^2 + 2SMF =$ (Prop. I.) $BA^2 + 2SMF = 4CA^2 + 2SMF$. Therefore $4CA^2 + 2SMF = 2CF^2 + 2CM^2$. Whence $CM^2 = 2CA^2 - CF^2 + SMF =$ (Prop. II.) $2CA^2 - CA^2 - CE^2 + SMF = CA^2 - CE^2 + SMF$.

Cor. *If* CA : CF : : CP : CI; *then rectangle* BPA = *rectangle* $SIF - MP^2$.

For $BPA = CP^2 - CA^2 = CM^2 - MP^2 - CA^2 = SMF - CE^2 - MP^2 =$ (Prop. VII.) $CI^2 - CA^2 - CE^2 - MP^2 =$ (Prop. II.) $CI^2 - CF^2 - MP^2 = SIF - MP^2$.

PROP. IX.

5. *If two lines* SM, FM, *be drawn from the foci, to any point* M *in the curve*; *they will make equal angles with the curve in that point.*

Take the point *m* infinitely near M in the curve, and draw S*m*, F*m*. And from S and F as centers, deſcribe the ſmall arches *mr*, *mt*, thro' *m*; cutting SM, FM in *r* and *t*. Then ſince $SM - FM = Sm - Fm$; therefore $SM + Fm = Sm + FM$; that is, $Sr + rM + Fm = Ft + tM + Sm$; therefore takeing away the equal quantities S*r*, S*m*, as alſo F*m*, F*t*; there remains $rM = tM$. Therefore in the very ſmall triangles M*mr*, M*mt*, right angled at *r* and t, we have $Mr = Mt$, and the hypothenuſe Fig. 5. Mm common; therefore the ſides mr, mt, are equal, and alſo the oppoſite angles, mMr, and mMt, or mMS and mMF are equal.

Cor. 1. *If* KMO *be drawn perp. to the tangent at* M; *then the angle* KMS = FMO.

For they are the complements of the equal angles mMS, and mMF.

Cor. 2. *If* MO *be drawn perp. to the tangent at* M, *and* SMD, FM, *drawn from the foci*; *the angle* FMO = DMO.

For DMO = KMS = FMO.

PROP. X.

6. *If* MP *be an ordinate to the tranſverſe axis, and* MO *perpendicular to the tangent at* M; S, F *the foci. Then* $CA^2 : CF^2 :: CP : CO$.

Make MG equal to MF, draw FG, which biſſect in D; and draw MD, which will be perp. to FG, and biſſect the angle GMF, and touch the curve at M, by Prop. IX. and ſince MO is alſo perp. to MD; therefore GF, MO are parallel; whence $SM : GM$ or $FM :: SO : FO$. And compounding and dividing, $SM+MF : SM-MF :: SO+OF : SO-OF$; that is, $SM+MF : 2CA :: 2CO : 2CF$. And $\frac{SM+MF}{2} : CA :: CO : CF$. And $\frac{SM+FM}{2} \times CA : CA^2 :: CO : CF$. But (Prop. V.) $\frac{SF+FM}{2} \times CA = CF \times CP$; therefore $CF \times CP : CA^2 :: CO : CF :: CF \times CO : CF^2$; and alternately, $CA^2 : CF^2 :: CF \times CP : CF \times CO :: CP : CO$.

Cor.

Fig. 6. Cor. 1. *If* CE *be the ſemiconjugate*, $CA^2 : CE^2 :: CP : PO$.

For $CA^2 : CF^2 :: CP : CO$; and by diviſion, $CA^2 : CF^2 - CA^2 :: CP : PO$. But (Cor. Prop. II.) $CF^2 - CA^2 = CE^2$; whence $CA^2 : CE^2 :: CP : PO$.

Cor. 2. BA : *latus rectum* : : CP : PO.

For CA : $\frac{1}{2}$ lat. rectum : : CA^2 : CA × $\frac{1}{2}$ lat. rectum or $CE^2 :: CP : PO$.

Cor. 3. $CF^2 : CE^2 :: CO : PO$.

For (this Prop.) $CF^2 : CA^2 :: CO : CP$. And (Cor. 1.) $CA^2 : CE^2 :: CP : PO$; therefore $CF^2 : CE^2 :: CO : PO$.

Cor. 4. $CA : CF :: MF : OF :: MS : OS$.

For we had before $SM + MF : 2CA :: CO : CF$; and by diviſion, $SM + MF - 2CA$ or $SM + MF - SM + MF : 2CA :: CO - CF$ or $OF : CF$; that is, $2MF : 2CA ::$ or $MF : CA :: OF : CF$. And from the firſt proportion by addition we have $SM + MF - 2CA : SM + MF + 2CA :: CO - CF : CO + CF$; that is, $2MF : 2MS :: OF : OS$.

PROP. XI.

6. *Let* S, F *be the foci*, MO *perp. to the tangent in* M; *then* $MO^2 = SO \times OF - SM \times MF$.

Draw the tangent MQ, and make MG = MF, and draw FG, which is perp. to MQ, becauſe the $\angle$ GMD = FMD. Therefore the triangles SGF, SMO are ſimilar, and $SO : FO :: SM : GM$ or $FM ::$ (Geom. II. 25.) $SQ : QF$. Whence $SO \times QF = SQ \times FO = SQ \times \overline{SO - SF}$, and $SQ \times SF = SO \times \overline{SQ - QF}$. But $SQ - QF = SQ - SQ - QF = 2SQ - SF = 2SQ - SO + FO$;

therefore $SO \times \overline{SQ - QF} = SO \times \overline{2SQ - SO + FO}$ Fig. 6. $= SQ \times SF = SQ \times \overline{SQ + QF}$, and $2QSO - SO^2 + SOF = SQ^2 + SQF$. And $SQ^2 + SO^2 - 2QSO + SQF = SOF$, or $\overline{SO - SQ}^2 + SQF = SOF$; that is, $QO^2 + SQF = SOF$. But (Geom. II. 26.) $QM^2 = SMF - SQF$. And in the right angled triangle QMO, $MO^2 = QO^2 - QM^2 = QO^2 - SMF + SQF = SOF - SMF$.

Cor. 1. *If* MQ *biſect the angle* SMF, *and* MO *be perpendicular to* MQ; *then* SQ : QF : : SO : OF. From the demonſtration.

Cor. 2. $CA^2 : CE^2 ::$ *rectangle* $SMF : MO^2$.
For we had SM : MF : : SO : OF; and by diviſion, SM — MF or 2CA : MF : : SO — OF, or 2CF : OF; and alternately,
CA : CF : : MF : OF, and
likewiſe CA : CF : : MS : SO; and multiplying,
$CA^2 : CF^2 :: SMF : SOF$,
and $CA^2 : CF^2 - CA^2$ or $CE^2 :: SMF : SOF - SMF$, or MO^2.

Cor. 3. *Rectangle* $SMF = BO \times OA - CE^2 - MO^2$.
For $SMF = SOF - MO^2$. But $SOF = CO^2 - CF^2 = CO^2 - CA^2 - CE^2 = \overline{CO + CA} \times \overline{CO - CA} - CE^2 = BO \times AO - CE^2$. Whence $SMF = BO \times OA - CE^2 - MO^2$.

Fig.

PROP. XII.

7. *If* SM, FM *be drawn from the foci to any point* M *in the curve, and* SG *be made equal to* BA, *and* FG *drawn; and if* MD *be drawn to the middle of* FG. *Then* MD *will touch the curve in* M.

For take any point m in the line MD, and draw Sm, Fm, Gm; then in the iſoceles triangles GmF, GMF, $Gm = mF$, and $GM = MF$. Now in the triangle SGm (Geom. II. 5.) $SG+Gm$ is greater than Sm; that is, $SG+Fm$ is greater than Sm, and Fm greater than $Sm - SG$; therefore the point m is without the curve. For if m was in the curve, and Sm, SG of given lengths, then $Fm = Sm - SG$, which now is bigger.

Cor. 1. *The tangent at* M *biſſects the angle* SMF, *made by two lines drawn from* M *to the focus.*

Cor. 2. *The tangents at the ends of the tranſverſe axis,* A *and* B, *are perp. to that axis.*

PROP. XIII.

8. *If* PM *be an ordinate,* DCE *the conjugate axis; then if* AD *be drawn to the ends of the tranſverſe and conjugate axes, and* PQ *parallel to it; then* $PM^2 = CQ^2 - CD^2 =$ *rect.* DQE.

For by ſimilar triangles $AC : CD :: PC : CQ$, and $AC^2 : CD^2 :: PC^2 : CQ^2$; and by diviſion, $AC^2 : CD^2 :: PC^2 - AC^2 : CQ^2 - CD^2$. But (Prop. VI.) $AC^2 : CD^2 ::$ BPA or $CP^2 - CA^2 : PM^2$; therefore $PM^2 = CQ^2 - CD^2 = DQE$.

PROP.

PROP. XIV.

If MP *be an ordinate to the tranſverſe axis,* F *the focus, and* HMO *be drawn perp. to the curve at* M, *cutting the conjugate* CD *in* H, *and the tranſverſe in* O. *Then will* $OP : OC :: CD^2 : CF^2$. 9.

For (Prop. X.) $CA^2 : CF^2 :: CP : CO$; and by diviſion, $CF^2 - CA^2 : CF^2 :: CO - CP : CO$, or $CD^2 : CF^2 :: PO : CO$.

Cor. 1. $CD^2 : CF^2 :: MO : OH :: MP : CH :: PO : CO$.
For the triangles MOP, HOC, are ſimilar.

Cor. 2. *If* L *be the latus rectum of* AB; *then As* $L : L + AB :: MO : OH :: OP : OC :: MP : CH$.
For (Cor. 1.) $CD^2 : CF^2 :: MO : OH$; but (Prop. IV. Cor. 1.) $CD^2 = CA \times \frac{1}{2} L$, and (Prop. II.) $CF^2 = CA^2 + CD^2 = CA^2 + CA \times \frac{1}{2} L$. Therefore $MO : OH :: CA \times \frac{1}{2} L :: CA^2 + CA \times \frac{1}{2} L :: \frac{1}{2} L : CA + \frac{1}{2} L :: L : BA + L$.

Cor. 3. $CA^2 : CD^2 :: CH - PM : PM :: CP : PO :: HM : MO$.
For (Cor. 1.) $CF^2 : CD^2 :: CH : MP$, &c. And by diviſion, $CF^2 - CD^2$ or $CA^2 : CD^2 :: CH - MP : MP :: CP : PO$, &c.

PROP. XV.

If PM *be an ordinate to the tranſverſe,* MQ *a tangent at* M, C *the center; then* CQ : CA : CP, *are continually proportional.* 6.

For ſince MQ biſſects the angle SMF (Prop. XII. Cor. 1.); therefore (Geom. II. 25.) SM : MF : :

Fig. 6. $MF : : SQ : QF$. And compounding and dividing, $SM+MF : SM-MF : : SQ+QF : SQ-QF$; that is, $SM+MF : 2AC : : 2CF : 2CQ$; and alternately, $AC : QC : : \frac{SM+MF}{2} : CF : : \frac{SM+MF}{2} \times AC : AC \times CF$. But (Prop. V.) $\frac{SM+MF}{2} \times AC = CF \times CP$. Whence $CA : CQ : : CF \times CP : CA \times CF : : CP : CA$, or $CP : CA : : CA : CQ$.

Cor. 1. *If* MQ *be a tangent at* M; C *the center*; MP *an ordinate*; *then* $BQ : AQ : : BP : AP$.

For $CP : CA : : CA : CQ$; and mixtly, $CP+CA : CP-CA : : CA+CQ : CA-CQ$; that is, $BP : AP : : BQ : AQ$.

Cor. 2. *If* MQ *be a tangent at* M, CE *the ſemi-conjugate*, MP *an ordinate*; *and* MO *perp. to* MQ. *Then* $CQ : CE : PO$, *are in geometrical progreſſion.*

For $CQ : CA : : CA : CP$, and $CQ^2 : CA^2 : : CA^2 : CP^2$. And (Prop. X. Cor. 1.) $CA^2 : CE^2 : : CP : PO$; and multiplying, $CQ^2 : CE^2 : : CA^2 \times CP : CP^2 \times PO : : CA^2 : CP \times PO : : CA^2 \times PO : CP \times PO^2$. But (ſame Cor.) $CA^2 \times PO = CE^2 \times CP$. Whence $CQ^2 : CE^2 : : CE^2 \times CP : PO^2 \times CP : CE^2 : PO^2$. And $CQ : CE : : CE : PO$.

Cor. 3. *All hyperbolas deſcribed on the ſame tranſverſe axis* AB, *will have their tangents to interſect at the ſame point of the axis* Q, *when the abſciſſa* AP *is the ſame.*

For if CP, CA remain the ſame; CQ is determined thereby, and muſt remain the ſame.

Cor. 4. *All tangents to the curve* AM, *interſect the axis* CA, *between the center* C, *and vertex* A.

PROP.

PROP. XVI.

All hyperbola's deſcribed on the ſame tranſverſe axis, will have their ordinates, ſtanding on the ſame abſciſſa, in a given ratio to one another; that is, as the reſpective conjugates. 4.

For (Prop. VI.) we have $BA^2 : NE^2 :: BPA : PM^2$; and alternately, $BA^2 : BPA :: NE^2 : PM^2$. Now if BA^2 and BPA remains the ſame, NE^2 and PM^2 will be in a given ratio, and PM to NE in a given ratio.

PROP. XVII.

If MT *be a tangent at* M, *and the line* FD *be drawn perpendicular upon it from either focus. The interſection* D *will be in the circumference of a circle* ADB, *deſcribed on the tranſverſe axis* BA, *as a diameter.* 10.

Draw SM from the other focus, and produce FD till it interſect it in G. Draw FM and CD. Then in the right angled triangles FMD, GMD; the angle FMD = (Cor. 1. Prop. XII.) GMD, and the ſide MD being common; therefore FD = DG, and FM = MG. And FC = CS, therefore CD is parallel to SG. Whence the triangles FCD and FSG are ſimilar. Therefore ſince $FD = \frac{1}{2} FG$, will $CD = \frac{1}{2} SG = \frac{SM - MG}{2} = \frac{SM - MF}{2} = \frac{1}{2} BA = CA$; therefore ſince CD = CA, the circle DAB whoſe radius is CA, will paſs thro' D.

And by a like reaſoning, if S be the other focus, and SH be perp. to MH, the point H will be in the circumference of the ſame circle DAB.

Cor. 1. *If a circle be deſcribed on the tranſverſe axis* AB *and a tangent at* M *cuts it in the points* D,

H,

Fig. H. *And* FD, SH *be drawn from the foci to* D *and*
10. H. *Then* FD, SH *are perp. to the tangent* HMD.

Cor. 2. *If* F, S *be the foci*; C *the center*; MD *a tangent at* M; *and if* CD *be drawn parallel to* SM, *to cut the tangent in* D. *Then* CD = CA *half the tranſverſe.*

Cor. 3. *If a tangent* MH *interſects the circle* DAHB *deſcribed on the tranſverſe, in* D *and* H; *the perpendiculars* DF, HS, *being drawn to it, will paſs thro' the foci,* F, S.

All theſe follow from the demonſtration of this Prop.

Cor. 4. *If* CI *be drawn parallel to the tangent at* M, *to interſect* SM, *then will* IM = *ſemitranſverſe* CA.

For then IM = CD = CA.

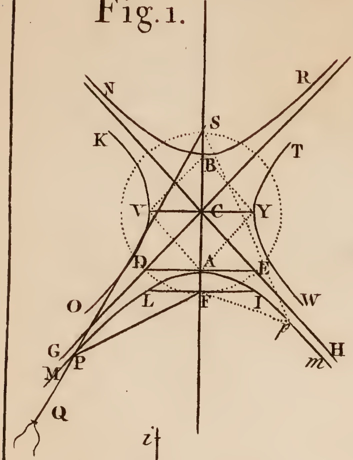
PROP. XVIII.

11. *If* F, S *be the foci,* C *the center,* MH *a tangent at* M. *If* SH, FD, *be drawn perp. to the tangent* MH. *Then* SH × FD = BS × SA = *the ſquare of the ſemiconjugate axis.*

About the tranſverſe deſcribe the circle DAHB, to interſect SH at R, and draw CR and CD. Then ſince the $\angle$ DHR is right, then RHD (Geom. IV. 14.) is a ſemicircle = BHA, therefore RB = DA, and $\angle$ RCB = DCA. Therefore in the triangles SCR, FCD; the ſides SC, CR are equal to FC, CD reſpectively; and the included angles at C are equal; therefore SR = DF. But (Geom. IV. 21.) HS × SR = AS × SB; that is, HS × DF = AS × SB = AF × FB = (Prop. III.) ſquare of the ſemiconjugate.

PROP.

Fig. 1.



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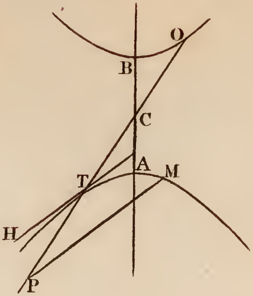

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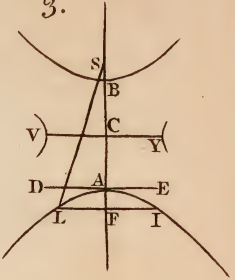

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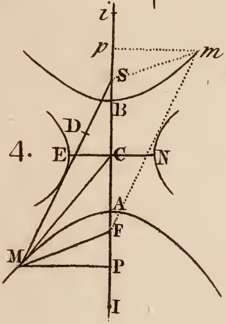

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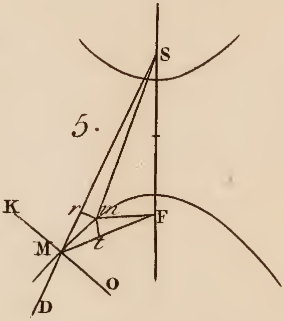

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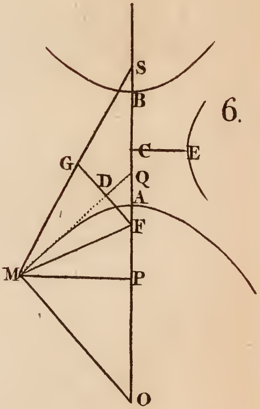

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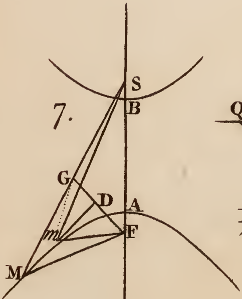

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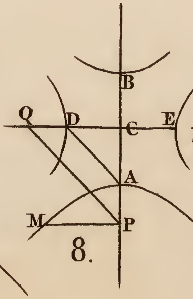

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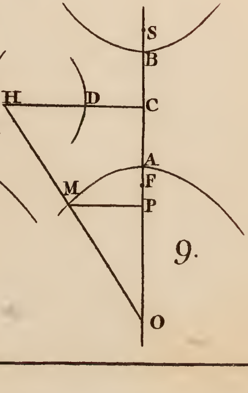

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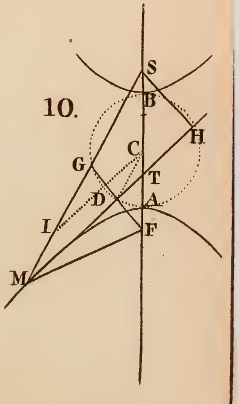


Fig.

PROP. XIX.

If MH *be a tangent at* M, *interſecting the axis* 11.
in F; FD, SH, *perpendiculars on it from the foci. Then* HM : MD : : HT : TD.

Deſcribe the circle BHAD about the tranſverſe AB, and draw SM, FM; then the triangles SMH, FMD are ſimilar; for the angles at H and D are right. And (Prop. IX.) SMH = FMD; therefore HM : MD : : HS : DF. And by the ſimilar triangles TSH, TFD, TH : TD : : HS : DF : : (before) HM : MD.

Cor. HM : MD : : CF + CT : CF — CT : : TS : TF.

For HM : MD : : TH : TD : : (ſimilar triangles) TS : TF.

PROP. XX.

If MH *be a tangent at* M, SH, FD *perpendiculars* 12.
on it from the foci; CE *the ſemiconjugate axis*; *then the rectangle* HMD = *rectangle* SMF — CE^2.

About the diameter BA deſcribe the circle DAHB, which will paſs through D and H (Prop. XVII.). Thro' the center C draw MO cutting the circle in I and O. Then (Geom. IV. 21.) HMD = OMI = $CM^2 - CI^2 = CM^2 - CA^2$. But (Prop. VIII.) $CM^2 - CA^2$ = SMF — CE^2; whence HMD = SMF — CE^2.

PROP. XXI.

If MH *be a tangent at* M; SH, FD *perpendi-* 12.
culars on it from the foci; CE *the ſemiconjugate. It will be* SM : MF : : $SH^2 : CE^2 : : CE^2 : FD^2$.

For (Prop. IX.) the angle SMH = FMD; therefore by ſimilar triangles, SM : MF : : SH : FD : : SH×FD :

Fig. $SH \times FD : FD^2$ (Prop. XVIII.) $:: CE^2 : FD^2$; and
12. $FM : MS :: FD : SH :: FD \times SH$ or $CE^2 : SH^2$.

PROP. XXII.

13. *If* MP *be an ordinate to the tranſverſe axis,* MH *a tangent at* M, C *the center,* AK, CG, BN *perp. to* AB. *Then* $AK : PM :: CG : BN$.

For (Prop. XV.) $TC : CA :: CA : CP$; and compounding, $TC : TC + CA$ or $TB :: CA : CA + CP$ or BP; and ſubtracting, $TC : TB :: CA - TC : BP - TB :: TA : TP$. But all the triangles TAK, TPM, TCG, TBN, are ſimilar; whence $TA : TP :: AK : PM$, and $TC : TB :: CG : BN$. Therefore $AK : PM :: CG : BN$.

Cor. *Hence* $TA : TP :: TC : TB$.

PROP. XXIII.

13. *If* PM *be an ordinate to the tranſverſe axis,* MH *a tangent at* M, C *the center,* CE *the ſemiconjugate;* AK, CG, BN *perpendiculars to* BA. *Then* $AK \times BN = PM \times CG = CE^2 = BSA = \frac{1}{4} BA \times$ *latus rectum.*

For (Prop. XXII.) $AK \times BN = PM \times CG$. Let fall SH, FD from the foci, perp. to MH. Then the triangles TBN, TSH, and alſo TAK, TDF are ſimilar; whence

$$TH : HS :: TB : BN$$
$$\text{and } TD : DF :: TA : AK$$

multiplying $TD \times TH : DF \times HS :: TA \times TB : AK \times BN$.

But (Prop. XVII.) the points D, H, are in the circumference of a circle deſcribed on the diameter AB. Therefore (Geom. IV. 20.) $TD \times TH =$

$TA \times TB$. Whence $DF \times HS = AK \times BN =$ Fig.
(Prop. XVIII.) CE^2. But (Prop. III.) $CE^2 = BSA$, 13.
and alſo (Prop. IV. Cor. 1.) $CE^2 = \frac{1}{2} BA \times \frac{1}{2}$ lat.
rectum.

Cor. 1. $FD \times SH = AK \times BN = CE^2$, *&c.*

Cor. 2. *The lines* NS, KS, *drawn to the focus, make a right angle* NSK.

For $NB : BS :: SA : AK$; therefore (Geom. II. 16.) the triangles NBS, SAK are ſimilar, and $\angle BNS = ASK$. But $\angle NSK = NSA + ASK = NSA + BNS = NBA =$ a right angle.

Cor. 3. *Hence a circle deſcribed on the diameter* NK, *will paſs thro' the foci,* S, F.

PROP. XXIV.

If S, F *be the foci,* MH *a tangent at* M, MO 14.
perp. to it, and OG *perp. to* SM; *then* MG *is half*
the latus rectum.

Draw SH, FD perp. to MH; then (Prop. XVIII.) $SH \times FD = CE^2$ the ſquare of half the conjugate. And ſince OG is perp. to MS, and OM perp. to MH; therefore $\angle GOM = HMS$; therefore the triangles SMH, FMD, and MOG are ſimilar; whence

$$MS : SH :: MO : MG$$
$$MF : FD :: MO : MG.$$

multiplying, $MF \times MS : SH \times FD :: MO^2 : MG^2$.

But (Prop. XVIII.) $SH \times FD = CE^2$ the ſquare of the ſemiconjugate; therefore $MF \times MS : CE^2 :: MO^2 : MG^2$; and $MF \times MS : MO^2 :: CE^2 : MG^2$. But (Prop. XI. Cor. 2.) $CA^2 : CE^2 :: MF \times MS : MO^2$; therefore $CA^2 : CE^2 :: CE^2 : MG^2$; and $CA : CE :: CE : MG$. Therefore (Prop. IV. Cor. 1.) MG is half the latus rectum.

Cor.

Fig. 14. Cor. 1. *If* MO *be perp. to the tangent at* M, L = *latus rectum. Then* AB+L : L : : *rectangle* SOF : MO^2.

For we had SMF : MO^2 : : CA^2 : CE^2 or CA × ½ L : : CA : ½ L; and compounding, SMF + MO^2 : MO^2 : : CA + ½ L : ½ L : : AB + L : L. But (Prop. XI.) SMF + MO^2 = SOF; therefore, &c.

9. Cor. 2. *If* MO *be perp. to the curve at* M, *cutting the two axes in* O *and* H; *and* S, F, *the foci. Then rectangle* SOF = MOH.

For SOF : MO^2 : : AB+L : L : : (Prop. XIV. Cor. 2.) HO : MO : : HO × MO : MO^2; therefore SOF = MOH.

Cor. 3. *Therefore a circle may be deſcribed from ſome point of the conjugate* CD, *which will paſs thro' all four points* F, S, H, M.

PROP. XXV.

15. *If* FH *be an ordinate at the focus,* TH *a tangent at* H, PM *any other ordinate, continued to the tangent at* G. *Then* FM = PG.

1. FH is = ½ latus rectum, and (Prop. XXIII.) FH × CN = CA × ½ latus rectum, therefore CN = CA.

2. Again, draw AL, BZ perp. to AB, and CD = ½ the conjugate; then (Prop. XV.) CA : CF : : CT : CA; and dividing, CF — CA or AF : CF : : CA — CT or TA : CA; and dividing again AF : (CF — AF or) AC : : TA : (CA — TA or) CT : : (ſimilar triangles) AL : CN or CA. Therefore AF = AL.

3. Likewiſe (Prop. XV. Cor. 1.) AF : FB : : TA : TB : : (ſimilar triangles) LA or AF : BZ. Therefore BZ = BF.

4. Alſo (Prop. XVI.) $CA^2 = CF \times CT = CF \times \overline{CF - TF} = CF^2 - CFT$, and $CF^2 - CA^2$ or $CD^2 = CF \times FT$. Fig. 15.

5. $TF : FH :: $ (ſim. triangles) $TC : CN$ or $CA :: TC \times CF$ or $CA^2 : CA \times CF :: CA : CF$; that is, $TF : FH :: CA : CF$.

6. By the laſt art. $CA : CF :: TF : FH ::$ (ſimilar triangles) $TP : PG$. And (Prop. XV. and VII. Cor. 3.) $CA : CF :: \left\{\begin{matrix} CT : CA \\ CP : CA + FM \end{matrix}\right\} ::$ (dividing) $CP - CT$ or $TP : FM$. Therefore $TP : PG :: TP : FM$; whence $PG = FM$.

From this proceſs are drawn the following corollaries.

Cor. 1. *If* HZ *be the focal tangent*, AL, CN, CD, BZ, PG, *perpendiculars on* AB; *then* $CN = CA$.

Cor. 2. $AL = AF$, *and* $BZ = BF$.

Cor. 3. $CF \times FT = CD^2$.

Cor. 4. $TF : FH :: CA : CF$.

Cor. 5 $TA : AF :: TP : FM$.
For $TA : AL$ (AF) $:: TP : PG$ (FM), by ſimilar triangles.

Cor. 6. $CA : CF :: TP : FM$.
For $CA : CF :: TF : FH :: TP : PG$ or FM.

PROP. XXVI.

16. *If* AB *be the tranſverſe axis,* C *the center,* F *the focus; and if* CF, CA, CT *be in geometrical progreſſion (or which is the ſame, if* TH *be the focal tangent), and* TE *be perp. to* TB. *Then if* MF *be drawn to the focus, and* ME *parallel to* TB; *it will always be* CF : CA : : FM : ME.

Draw the ordinate FH at the focus, and the tangent TH; then (Prop. XV.) CF : CA : : CA : CT; and therefore (Prop. XXV. Cor. 6.) CF : CA : : FM : TP, or CF : CA : : FM : EM.

Cor. MF *is to* ME *always in the ſame given ratio, wherever the point* M *is taken.*

SCHOLIUM.

The writers on Conic Sections, call the line TE, the *Directrix.*

PROP. XXVII.

17. *Any diameter* MN *is biſſected in the center* C.

For take CP = CQ, and erect the ordinates PM, QN; and draw NM; then (Cor. 5. Prop. VI.) QN = MP, therefore the triangles CPM, CQN are equal and ſimilar; therefore CN = CM.

PROP. XXVIII.

18. *If* DF *be an hyperbola, whoſe center is* C, *and ſemitranſverſe* CD, *ſemiconjugate* CA, QV *parallel to* CD. *Then* $CA^2 : CD^2 :: CA^2 + CQ^2 : QV^2$.

Draw the ordinate BV; then (Prop. VI.) $CA^2 : CD^2 :: BV^2 : CB^2 - CD^2$; and by addition, $CA^2 :$ $CD^2 ::$

$CD^2 :: CA^2 + BV^2 : CB^2$; that is, $CA^2 : CD^2 :: CA^2 + CQ^2 : QV^2$. Fig. 18.

Cor. 1. *If* AM *be its conjugate hyperbola,* PM *an ordinate; then* $CP^2 - CA^2 : PM^2 :: CA^2 + CQ^2 : QV^2$.

For $CP^2 - CA^2 : PM^2 :: CA^2 : CD^2 ::$ (this Prop.) $CA^2 + CQ^2 : QV^2$.

Cor. 2. $CA^2 + CQ^2 : QV^2 :: CA^2 + Cq^2 : qv^2$.

PROP. XXIX.

If MC, VC *be two conjugate ſemidiameters; and if from the ends* M, V, *the perpendiculars* MP, VQ *be let fall on the tranſverſe* AB. *Then* $CQ^2 = AP \times PB = CP^2 - CA^2$. 19.

For draw the tangent MT, and (Prop. XV.) $CP : CA :: CA : CT$; and dividing, $CP : AP :: CA : AT ::$ (by adding) $BP : PT$. Whence $CP \times PT = AP \times PB = CP^2 - CA^2$, and $CA^2 = CP^2 - CPT$.

The tangent MT (def. 7.) is parallel to CV; whence the triangles TPM and CQV are ſimilar. $AC^2 + CQ^2 = CP^2 - CPT + CQ^2$. And (Cor. 1. Prop. XXVIII.) $APB : CA^2 + CQ^2 :: PM^2 : QV^2 ::$ (ſim. triangles) $TP^2 : CQ^2$. That is, $CP \times PT : CP^2 + CQ^2 - CPT :: TP^2 : CQ^2$; and alternately, $CPT : TP^2 :: CP^2 + CQ^2 - CPT : CQ^2$; and ſubtracting, $CPT - TP^2 : TP^2 :: CP^2 - CPT : CQ^2$; that is, $\overline{CP - TP} \times PT : TP^2 : \overline{CP - TP} \times CP : CQ^2$, or $PT : PC :: TP^2 : CQ^2$. Whence $PT \times CQ^2 = PC \times TP^2$, and $CQ^2 = TP \times PC = AP \times PB$.

Cor. 1. $CP^2 = CA^2 + CQ^2 = CA^2 + APB$.

Cor. 2. $CP^2 - CQ^2 = CA^2 = CP^2 - APB$.

Fig. Cor. 3. $QV^2 - PM^2 = CD^2$.

19. For (laſt Prop.) $CA^2 : CD^2 :: CA^2 + CQ^2$:
QV^2 : : (Cor. 1. Prop. XXVIII.) $CP^2 - CA^2$ or
$CQ^2 : PM^2$: : (ſubtracting the antecedents) CA^2 :
$QV^2 - PM^2$; therefore $CD^2 = QV^2 - PM^2$.

Cor. 4. $CA : CD :: CP : QV :: CQ : PM$.
For (Prop. XXVIII. and Cor.) $CA^2 : CD^2$:
($CA^2 + CQ^2$ or) $CP^2 : QV^2$: : ($CP^2 - CA^2$ or
$CQ^2 : PM^2$.

PROP. XXX.

19. *The difference of the ſquares of two conjugate diameters is equal to the difference of the ſquares of th two axes*, $CM^2 - CV^2 = CA^2 - CD^2$.

For in the right ∠ triangles CPM, CQV, CM
$= CP^2 + PM^2$, and $CV^2 = CQ^2 + QV^2$; therefore $CM^2 - CV^2 = CP^2 - CQ^2 + PM^2 - QV$
$=$ (XXIX. Cor. 2.) $CA^2 + PM^2 - QV^2 =$
(Cor. 3.) $CA^2 - CD^2$.

Cor. *In an equilateral hyperbola, every diameter equal to its conjugate.*

PROP. XXXI.

19. *Let* MC, CV *be two conjugate ſemidiameters*; *th the rectangle of the diſtances of the foci from the vertex* M *of the diameter* MC, *is equal to the ſquare the ſemiconjugate* CV, $FM \times SM = CV^2$.

Let CA, CD be the ſemiaxes Then (Prop. VIII
$CM^2 = CA^2 - CD^2 + SMF$, and SMF
$CM^2 - CA^2 + CD^2 =$ (Prop. XXX.) CM^2
$CV^2 - CM^2 = CV^2$.

Cor. 1. *If* F, S, *be the foci*; *then* $FM \times SM$
$MC \times \frac{1}{2}$ *lat. rectum of* MC.

F

For $CV^2 = MC \times \frac{1}{2}$ lat. rectum of MC. Fig. 19.

Cor. 2. *If* MO *is perp. to the tangent at* M; *then* CA : CD : : CV : MO.

For (Prop. XI. Cor. 2.) $CA^2 : CD^2 :: SMF$ or $CV^2 : MO^2$; and CA : CD : : CV : MO.

PROP. XXXII.

If F, S, *be the foci*, CM, CK *ſemidiameters con-* 20.
iugate to one another, TM *a tangent at any point* M; FT, CI, SH *perpendiculars on the tangent* TM; CD *the ſemiconjugate axis. Then* FM : FT : : SM : SH : : CK : CD.

For ſince CF = CS, FT — SH = 2CI. The triangles FMT, SMH, are ſimilar; whence MF : FT : : MS : SH : : (MF — MS) 2AC : (FT — SH) 2CI : : AC : CI; that is,

AC : CI : : MF : FT, and
ſimilar triangles AC : CI : : MS : SH;
therefore $AC^2 : CI^2 :: MS \times MF : SH \times FT$.
But (XXXI.) $MS \times MF = CK^2$, and (XVIII.) $SH \times FT = CD^2$; therefore,

$$AC^2 : CI^2 :: CK^2 : CD^2 :: MF^2 : FT^2.$$

and AC : CI : : CK : CD : : MF : FT : : MS : SH.

Cor. MF : FT : : AC : CI : : CK : CD.
From the demonſtration.

PROP. XXXIII.

All inſcribed parallelograms, whoſe ſides are paral- 21.
el to two conjugate diameters, are equal to the rectan-le of the two axes.

Let CI be perp. to the ſide GH, that touches he curve in M; then (Cor. Prop. XXXII.) AC :

CI : :

Fig. CI : : CK : CD ; therefore CK × CI = AC × CD ;
21. but CK × CI = KCMG = AC × CD. But KCMG
is a quarter of the inſcribed parallelogram ; and AC
× CD is a quarter of the rectangle of the axes ;
therefore the wholes are equal.

PROP. XXXIV.

22. *Let* CA *be the ſemitranſverſe axis* ; CM *any ſemidiameter* ; AE, MT, *two tangents at* A *and* M. *Then the triangle* CAE = CTM.

Draw the ordinates MP, AD, to the axis CA and diameter CM. Then the triangles CPM, CAE and alſo CAD, CTM, are ſimilar. Whence CD CM : : CA : CT : : (XV.) CP : CA : : CM : CE Therefore (Geom. II. 12. Cor. 1.) the lines DP MA, ET, are all parallel to one another. Whenc (Geom. II. 10.) triangle AME = triangle AMT which taken from the triangle CAM, leaves triangle CAE = triangle CMT.

Cor. 1. *The lines* DP, MA, ET, *are parallel t one another.*

This is plain from the demonſtration.

Cor. 2. *The triangle* AIT = *triangle* MIE.

This appears by taking CTIE from the equ triangles CAE, CTM.

Cor. 3. *The triangle* MPT = *triangle* ADE = *trapezoid* MPAE = *trapezoid* MDAT.

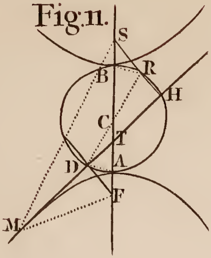
For AME = AMT, add the triangle AMP then PMT = MPAE ; or add AMD (= AMP then ADE = (MPT =) ADMT.

Cor. 4. *Triangle* CDA = CPM.

For CD : CM : : CP : CA. Therefore (Geom. I 17. Cor. 1.) CDA = CPM.

PRO

Fig. 11.



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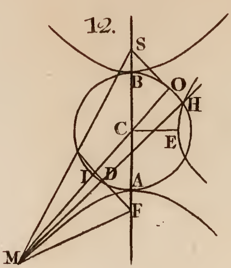

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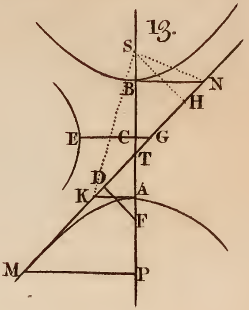

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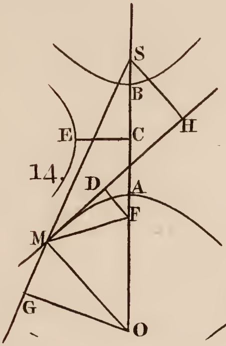

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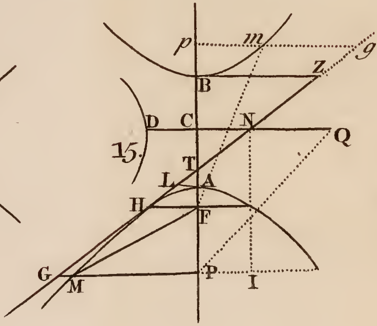

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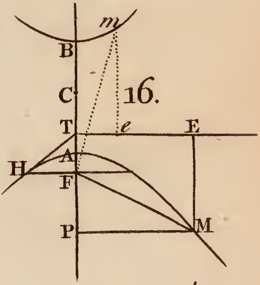

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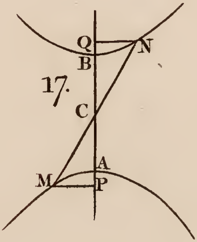

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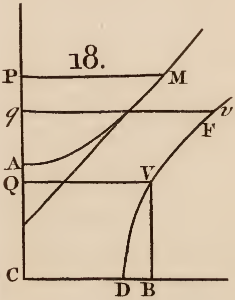

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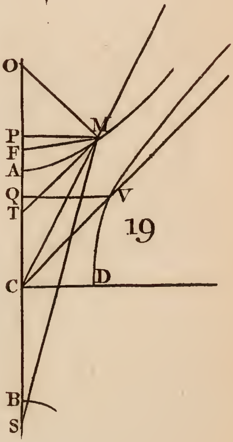


Pl.XIV.*pa.104*

PROP. XXXV.

Fig. 23. *Let* BA *be the tranſverſe,* MV *a diameter,* MP, QL *ordinates to them*; AE, MT, *tangents*; QFI *perp. to* AB. *Then the triangle* QIR = *trapezoid* IAEF.

Let CK be the ſemiconjugate to CM. Produce QL to R. By ſimilar triangles, CA : AE : : CP : PM : : CI : IF. And by adding the antecedents, and the conſequents, CA : AE : : CA + CP : AE + PM : : CA + CI : : AE + IF. But the rectangle BIA = $AI \times \overline{CA + CI}$, and rectangle BPA = $AP \times \overline{CA + CP}$. Therefore BPA : BIA : : $AP \times \overline{CA + CP} : AI \times \overline{CA + CI} : : AP \times \frac{AE + PM}{2} : AI \times \frac{AE + IF}{2}$: : trapezoid PAEM : AEIF.

The triangles MPT, QIR are ſimilar; whence triangle MPT : QIR : : $MP^2 : QI^2$: : BPA : BIA : : trapezoid PAEM : trapezoid AEIF. But (XXXIV. Cor. 3.) MPT = PAEM, therefore QIR = AEIF.

Cor. *The triangle* QLF = *trapezoid* LMTR. For (XXXIV.) CAE = CMT, ſubtract both from CLR, and then LRAE = LMTR. But LRAE = FIAE — FIRL = QIR — FIRL = QLF; therefore LMTR = QLF.

PROP. XXXVI.

23. *Let* CK *be the ſemiconjugate to* MC; LQ *an ordinate to* MC; *then* $CM^2 : CK^2$: : *rectangle* MLV : LQ *ſquare.*

Let BA be the tranſverſe axis, produce it, and draw KO perpen. to it, to cut CM in O. Then (Cor. XXXV.) QLF = LMTR.

But

Fig. 23. But (XXIX.) $CP^2 - CA^2 = CN^2$, and (ib. Cor. 4.) $CP \times PM = CN \times NK$, therefore triangle CPM = triangle CNK. Alſo the triangle $CPM : CAE :: CP^2 : CA^2$. And by diviſion, triangle CPM : ($CPM - CAE$ or) PAEM :: CP^2 : ($CP^2 - CA^2$ or) CN^2 :: triangle CPM : triangle CNO; therefore PAEM = CNO, and ſubtracting theſe from the equal triangles CPM and CNK, and then CAE = COK, or (XXXIV.) CTM = COK.

The triangle $CTM : CRL :: CM^2 : CL^2$; and $CTM : CRL - CTM :: CM^2 : CL^2 - CM^2$; that is, triangle CTM : trapezoid LMTR : CM^2 : rectangle VLM. But the triangles OKC, FQL, are ſimilar; therefore $CK^2 : LQ^2$ :: triangle OKC : triangle FQL; that is, $CK^2 : LQ^2$ :: CMT : LMTR :: (before) CM^2 : rectangle VLM; or $CK^2 : CM^2 :: LQ^2$: rectangle VLM.

Cor. 1. *As the tranſverſe* MV : *to its latus rectum* :: *ſo the rectangle* MLV : *to ſquare of the ordinate* LQ^2.

Cor. 2. *The rectangles of the diſtances from the vertexes of any diameter, are as the ſquares of the ordinates, ſtanding at theſe diſtances.*

Cor. 3. *The diameter biſſects all the double ordinates. And the ordinates at equal diſtances from the vertexes, are equal.*

For the ratio being the ſame for the ordinates on each ſide, ſtanding on the ſame point; theſe ordinates muſt be equal. And as the terms of the ratio remain the ſame at equal diſtances from the vertex; therefore the ordinates will be equal at equal diſtances.

Cor. 4. *Triangle* CNO = *trapezoid* APME, *and* CNK = CPM, *and* COK = CTM.

PROP.

Fig.

PROP. XXXVII.

If two right lines GI, HP *be drawn parallel to* 24.
two conjugate diameters DK, MV, *to interſect in* R; 25.
then $CM^2 : CK^2 ::$ *rectangle* PRH : *rectangle* GRI.

For (XXVIII. Cor. 2.) $PN^2 : IO^2 :: CK^2 + CN^2 : CK^2 + CO^2$. And (fig. 24.) $PN^2 : IO^2 - PN^2 :: PK^2 + CN^2 : CO^2 - CN^2$. But $IO^2 - PN^2 = RN^2 - PN^2 = PRH$. And $CO^2 - CN^2 = LI^2 - LR^2 = GRI$.

Alſo (fig. 25.) $PN^2 : PN^2 - IO^2 :: CK^2 + CN^2 : CN^2 - CO^2$. But $PN^2 - IO^2 = PN^2 - RN^2 = PRH$. And $CN^2 - CO^2 = RL^2 - LI^2 = RL^2 - LG^2 = GRI$. Whence $PN^2 : PRH :: CK^2 + CN^2 : GRI$. And alternately $PRH : GRI :: PN^2 : CK^2 + CN^2 ::$ (Prop. XXVIII.) CV^2 or $CM^2 : CK^2$.

Cor. *Hence if more lines be drawn parallel to* HP, GI; *the rectangles of their ſegments, will be as the rectangles* PRH *and* GRI, *parallel to them. The ſame holds in the oppoſite hyperbola, with the ſmall letters.*

PROP. XXXVIII.

If a line HG *interſect any diameter* AB *in* D, *and* 26.
a diameter FK *be drawn parallel to* HG. *Then,* 27.
rectangle ADB : *rectangle* $GDH :: CB^2 : CF^2$.

Draw the diameter MV conjugate to FK, and AI parallel to GH. Then the triangles CDE, CAI are ſimilar; therefore $CI : CA :: CE : \dfrac{CA \times CE}{CI} = DC$; and $CI : AI :: CE : DE = \dfrac{AI \times CE}{CI}$.

Alſo

Fig. 26. 27. Alſo (Prop. XXXVI.) $CM^2 : CF^2 :: CE^2 - CM^2 : EH^2 = \frac{CE^2 \times CF^2}{CM^2} - CF^2$. Hence (fig. 26.) ADB $= CD^2 - CA^2 = \frac{CA^2 \times CE^2}{CI^2} - CA^2 = CA^2 \times \frac{CE^2 - CI^2}{CI^2}$.

Again $CF^2 : CM^2 :: AI^2 : CI^2 - CM^2$, and $CM^2 \times AI^2 = CF^2 \times CI^2 - CF^2 \times CM^2$; and $CF^2 \times CI^2 - CM^2 \times AI^2 = CF^2 \times CM^2$.

But rectangle GDH $= EH^2 - ED^2 =$

$$\frac{CE^2 \times CF^2}{CM^2} - CF^2 - \frac{AI^2 \times CE^2}{CI^2} =$$

$$\frac{CE^2 \times CF^2 \times CI^2 - AI^2 \times CE^2 \times CM^2}{CM^2 \times CI^2} - CF^2 =$$

$$\frac{CE^2 \times CF^2 \times CM^2}{CM^2 \times CI^2} - CF^2 = \frac{CE^2 \times CF^2}{CI^2} - CF^2 =$$

$$CF^2 \times \frac{CE^2 - CI^2}{CI^2}.$$

Therefore rectangle ADB : GDH : : $CA^2 : CF^2$.

And (fig. 27.) the demonſtration is the ſame, only ADB $= CA^2 - CD^2$, and GDH $= ED^2 - EH^2$. Whence the ſame concluſion will follow.

Cor. *The ſame holds in the oppoſite hyperbola, with the ſmall letters.*

PROP. XXXIX.

28. *If two lines* GH, LE *interſect one another in* D *(within or without the curve), and the ſemidiameters* CF, CK *be drawn parallel to them; the rectangle* GDH : *rectangle* LDE : : $CF^2 : CK^2$.

Thro' D draw the ſemidiameter CB; then (laſt Prop.) rectangle GDH : rect. ADB : : $CF^2 : CB^2$;

and rectangle ADB : LDE : : $CB^2 : CK^2$; therefore *(ex equo)* GDH : LDE : : $CF^2 : CK^2$. Fig. 28.

And a like demonſtration is had when the interſection is without the curve.

Cor. 1. *If any line* GH *cut two parallels* LE, MQ, *in* D *and* N, *then* HDG : EDL : : HNG : QNM.

For EL, QM being parallel to the ſame diameter; theſe rectangles will be as the ſquares of the ſemidiameters parallel to theſe lines.

Cor. 2. *If two parallel lines* GH, TO, *cut other two parallels* LE, MQ; *their ſeveral rectangles will be in the ſame ratio*; HDG : EDL : : HNG : QNM : : ORT : ERL : : OST : QSM; *and that whether they interſect within or without the curve. And in the ſame, or the oppoſite hyperbolas.*

For each pair will be reſpectively proportional to the ſquares of the ſemidiameters, parallel to theſe lines.

PROP. XL.

29. *If* DM *be a tangent at* M, *and* DN *any line cutting the hyperbola (or the oppoſite ſections) in* O *and* N; *and if* CF, CK *be ſemidiameters parallel to* DM, DN. *Then* DM^2 : *rectangle* ODN : : $CF^2 : CK^2$.

For (by Prop. XXXVIII. and fig. 27.) we have ADB : GDH : : $CB^2 : CF^2$. And ſuppoſing G, H, to coincide in M; we ſhall have (fig. 29.) ADB : DM^2 : : $CB^2 : CF^2$. And alternately, ADB : CB^2 : : $DM^2 : CF^2$. And (by the ſame Prop.) ADB : CB^2 : : ODN : CK^2; therefore $DM^2 : CF^2$: : ODN : CK^2.

30. Cor. 1. *If* DM, DN *be two tangents, and the ſemidiameters* CF, CK *be drawn parallel thereto; then* DM : DN : : CF : CK.

For

Fig. 30. For ſuppoſe O, N, to come together in one point (fig. 29.) then ODN, becomes DN^2; whence $DM^2 : DN^2 :: CF^2 : CK^2$, and DM : DN :: CF : CK.

31. Cor. 2. *If two parallel tangents* DM, TG, *meet a third tangent* NDT; *then* DM : DN :: TG : TN.

For if the ſemidiameters CK, CF be drawn parallel to DT and DM. It will be DM : DN :: FC : CK, and TG : TN :: CF : CK. Therefore DM : DN :: TG : TN.

32. Cor. 3. *If* M, G *be the points of contact of the parallel tangents* MD, GT; *and* N *the point of contact of the tangent* ND. *And if* OE, CF *be parallel to* DM, *and* QR, CK *parallel to* ND. *Then* $CF^2 : CK^2 :: DM^2 : DN^2 :: TG^2 : TN^2 :: IOE : NO^2 :: IQE : PQL :: GR^2 : LRP$.

All this appears from this Prop. and Cor. 1. and 2.

PROP. XLI.

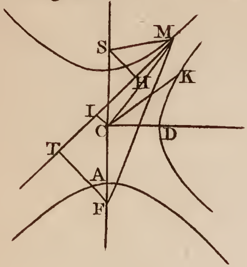
33. *If* DM, DN *touch the hyperbola at* M *and* N, *and* MN *be drawn, and any line* RL *parallel to* DN; *cutting* DM, MN, *in* G *and* H; *then* GR, GH, GL *are continually proportional.*

For (Cor. 3. laſt Prop.) $RGL : GM^2 :: DN^2 : DM^2 ::$ (ſimilar triangles) $GH^2 : GM^2$. Therefore $RGL = GH^2$; and GR : GH : GL $\div$.

Cor. *If* FP, *parallel to* DN, *touch the curve in* P, *and cuts* DM *in* O; *then* OP = OF.

For then GL will fall on FP; the points R and L upon P, G upon O, and H upon F. Whence GR = GL = OP, and FO = GH. Therefore OP^2 (RGL) = OF^2 (GH^2), and OP = OF.

Fig. 20.



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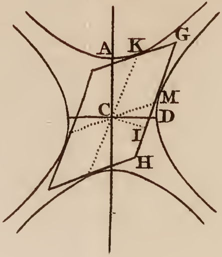

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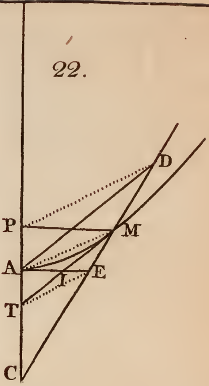

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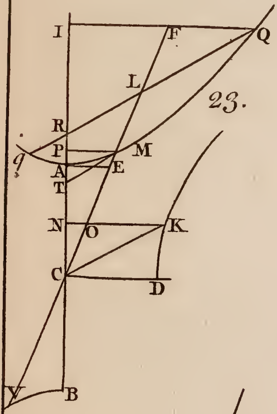

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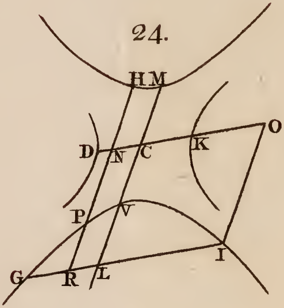

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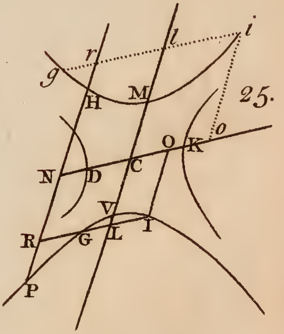

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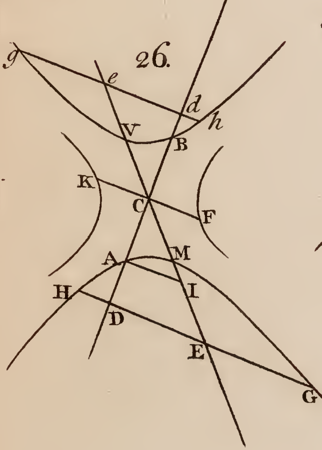

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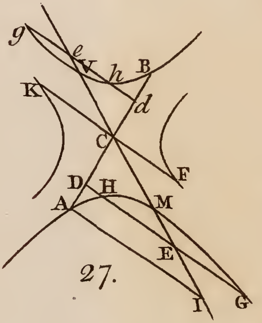

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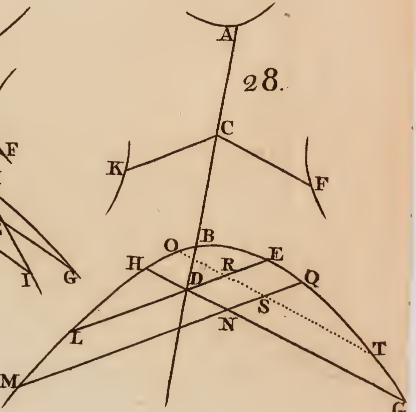


Pl. XV. pa.11

PROP. XLII.

If AB *be any diameter,* MT *a tangent cutting it in* T, *and* MP *an ordinate. Then* CT, CA, CP *are continually proportional.* 34.

At A and B draw the tangents AK, BH, cutting MT in K and H; then ſince AK, PM, BH are parallel; BP : AP : : HM : MK : : (Cor. 2. XL.) HB : AK : : (ſimilar triangles) TB : TA. And BP + AP : BP : : TB + TA : TB, or 2CP : BP : : 2CB : TB, and CP : BP : : CB : TB. And dividing PC : BP — PC : : CB : TB — CB; that is, PC : BC : : BC : CT.

Cor. 1. *If* MT *be a tangent at* M, MP *an ordinate. Then* BP : PA : : BT : AT. *And* BP *is harmonically divided in* T *and* A.

From the demonſtration.

Cor. 2. *If* MT *be a tangent at* M, MP *an ordinate. Then* PA : PT : : PC : PB.

For we had PC : PB : : BC : BT; and by diviſion, PC : PB : : (PC — BC or) PA : (PB — BT or) PT.

Cor. 3. *If* MT *be a tangent at* M, MP *an ordinate. Then* TA : TP : : TC : TB.

For CT : CA : : CA : CP; and dividing, CA — CT : CT : : CP — CA : CA, or TA : TC : : AP : AC; and adding, TA : TC : TA + AP : TC + AC. That is, TA : TC : : TP : TB.

Cor. 4. *Two tangents drawn from the ſame double ordinate, on each ſide the curve, meet in one and the ſame point of the diameter.*

For the ſame quantities, and the ſame proportion, determine both of them.

Fig. 34. Cor. 5. *If* AB, DS *be two conjugate diameters,* MR *parallel to* AB, *cutting* DS *in* R; MG *a tangent at* M, *cutting the conjugates in* T *and* G. *Then* $CR : CS :: CS : CG$.

Draw the ordinate PM, then by ſimilar triangles, $CG : CT :: RG : RM$ or $CP ::$ (by diviſion) $CR : TP$; and alternately, $CG : CR :: CT : TP$. But (XXXVI.) $CS^2 : CA^2 :: PM^2 : BPA$; that is, $CS^2 : CP \times CT :: CR^2 : CP^2 - CA^2$ or $CP^2 - CP \times CT$; alternately, $CS^2 : CR^2 :: CP \times CT : CP^2 - CP \times CT$ or $CP \times TP :: CT : TP ::$ (before) $CG : CR :: CG \times CR : CR^2$. Therefore $CS^2 = CG \times CR$.

Cor. 6. *The ſame things ſuppoſed as in Cor. 5. the rectangle* $GRC = CS^2 + CR^2$, *and rectangle* $RGC = CS^2 + CG^2$.

For $CS^2 = CG \times CR = \overline{GR - CR} \times CR = GR \times CR - CR^2$, and $GR \times CR = CS^2 + CR^2$.

Again $CS^2 = CG \times CR = CG \times \overline{RG - CG} = CG \times RG - CG^2$, and $RG \times CG = CS^2 + CG^2$.

PROP. XLIII.

35. *Let* AB *be any diameter*, CD *the ſemiconjugate*, AM, DN, *conjugate hyperbola's*, PM *an ordinate*, QV *parallel to* CD. *Then* $CA^2 : CD^2 :: CA^2 + CQ^2 : QV^2$.

Draw the ordinate BV parallel to CA. Then (Prop. XXXVI.) $CA^2 : CD^2 :: BV^2 : CB^2 - CD^2$; and compounding, $CA^2 : CD^2 :: CA^2 + BV^2 : CB^2$. That is, $CA^2 : CD^2 :: CQ^2 + CA^2 : QV^2$.

Cor. 1. *If* AM, DN *be conjugate hyperbolas*; CA, CD *conjugate ſemidiameters*, PM *an ordinate*, QV *parallel to it*. *Then* $CP^2 - CA^2 : PM^2 :: CQ^2 + CA^2 : QV^2$.

For $CP^2 - CA^2 : PM^2 :: CA^2 : CD^2 :: CA^2 + CQ^2 : QV^2$. Fig. 35.

Cor. 2. *If* qv *be parallel to* QV; *then* $CQ^2 + CA^2 : QV^2 :: Cq^2 + CA^2 : qv^2$.

Cor. 3. *If the ordinate* PM *be produced to the conjugate hyperbola at* N. *Then* $CP^2 - CA^2 : PM^2 :: CP^2 + CA^2 : PN^2$.

For $CP^2 - CA^2 : PM^2 :: CA^2 : CD^2 :: CA^2 + CP^2 : PN^2$.

Cor. 4. $PN^2 + PM^2 : PN^2 - PM^2 :: CP^2 : CA^2$. By compounding the terms in Cor. 3.

PROP. XLIV.

If AB *be any diameter*, CD *its ſemiconjugate*; HT *a tangent at* M, *cutting the parallel tangents* AK, BH, *in* K *and* H; MP *an ordinate. Then* AK, CD, BH, *are in continual proportion.* 34.

By ſimilar triangles $AK : PM :: TA : TP ::$ (XLII. Cor. 3.) $TC : TB ::$ (ſim. triangles) $CG : BH$. Whence $PM \times CG = AK \times BH$.

Then $CP : CT :: CP^2 : CP \times CT$ or CA^2 (XLII.). And by diviſion, $CP - CT$ or $PT : CT :: CP^2 - CA^2$ or $BPA : CA^2 ::$ (XXXVI.) $PM^2 : CD^2$. And by ſimilar triangles, $TP : TC :: PM : CG :: PM^2 : PM \times CG$ or $AK \times BH$; therefore $PM^2 : CD^2 :: PM^2 : AK \times BH$. Whence $AK \times BH = CD^2$.

Cor. 1. *If* AB *be any diameter*, CD *its ſemiconjugate*, PM *an ordinate*; AK, BH, HM, *tangents at* A, B, M; *then* $AK \times BH = PM \times CG = CD^2 = AC \times \frac{1}{2}$ *latus rectum of* AB.

For we had before $AK \times BH = PM \times CG =$ (by this Prop.) $CD^2 =$ (Schol. IV.) $\frac{1}{4} AB \times$ latus rectum.

Cor.

Fig. 36. Cor. 2. *If the two parallel tangents at the vertexes,* AK, BH, *be cut by the two tangents* MKH, QRN; *then* AK × BH = BQ × AR.

For they are both equal to CD^2.

Cor. 3. *The ſame things ſuppoſed,* AK : BQ : : KO : OH.

For AK : BQ : : AR : BH : : AK + AR or KR : BQ + BH or QH : : (ſimilar triangles) KO : OH.

Cor. 4. *A line drawn thro'* RH, *would interſect the line drawn thro'* KQ, *ſomewhere in the diameter* AB *produced.*

This is plain from the proportions in Cor. 3.

Cor. 5. *If the tangent* QR *was to touch the oppoſite hyperbola (as qr), it will ſtill be* AK × BH = *Bq* × A*r*.

PROP. XLV.

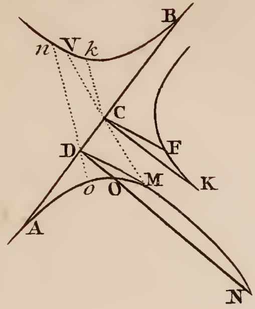
37. *In the hyperbola and oppoſite ſections if* TH, TK, *be two tangents at* H *and* K; *and if* HK *be drawn thro' the points of contact, and biſſected in* O, *and if* TO *be drawn; it will biſſect all the parallels to* HK, *as* IN. *That is, if* HO = OK, *then* IP = PN.

Thro' the center C draw FG parallel to HK; then ſince HO = OK, CF = CG. And CA is a ſemidiameter, and HO, OK, ordinates to it. And ſince IN is parallel to HK; IP, PN will alſo be ordinates; and (XXXVI. Cor. 3.) IP = PN; and CGE and CFD are ſemiconjugates.

Cor. 1. *If* LM *be drawn parallel to* HK, *which paſſes thro' the points of contact; the parts intercepted between the curve and tangent, on each ſide, are equal;* LI = MN.

For

Fig: 29.



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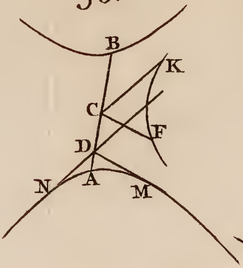

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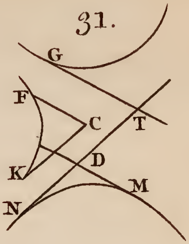

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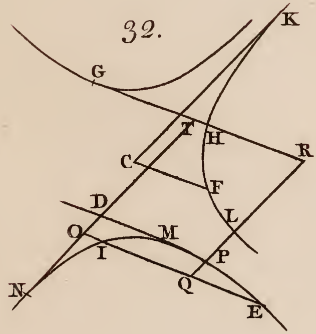

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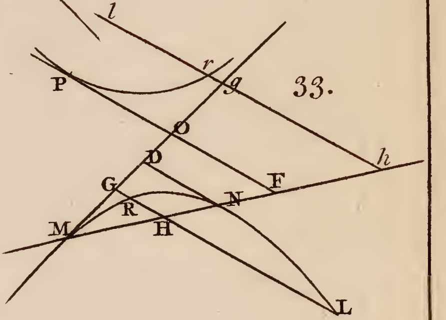

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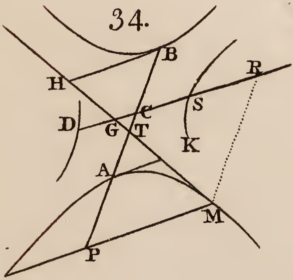

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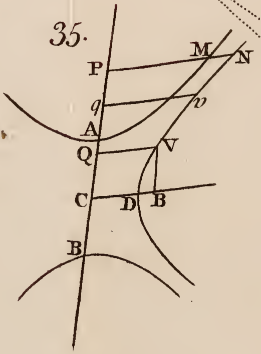

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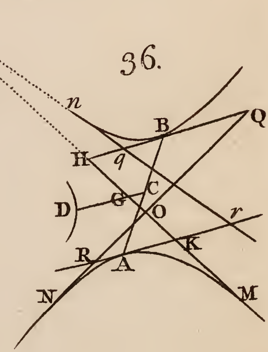


Pl. XVI. *pa. 114.*

For ſince HO = OK, therefore LP = PM, and and IP being = PN, the remainder IL = NM. Fig. 37.

Cor. 2. *The tangent at* A, *is biſſected in the point* A.

Cor. 3. HO, OK, *and alſo* IP, PN, *are ordinates to the diameter* CA, *paſſing thro'* T *the interſection of the tangents.*

SCHOLIUM.

All theſe things hold equally, if one of the tangents be drawn to the oppoſite hyperbola.

PROP. XLVI.

If TL, TM *be two tangents at* H *and* K, *and* HK *be drawn thro' the points of contact; and from the interſection* T, *any line* TI *be drawn, cutting* HK *in* D, *and the curve in* F *and* I. *Then* TF : TI : : DF : DI. 38.

Thro' F, I, draw AC, LM, parallel to HK; then (Prop. XLV. Cor. 1.) LI = NM, and AF = BC; whence MI = LN, and CF = AB.

by ſimilar triangles, AF : LI : : TF : TI.

and CF : MI : : TF : TI.

multiplying, $AF \times CF :: LI \times MI :: TF^2 : TI^2$.

that is, $AF \times AB :: LI \times LN :: TF^2 : TI^2$.

But (Prop. XL. Cor. 3.) $AF \times AB :: LI \times LN :: HA^2 : HL^2 ::$ (ſimilar triangles) $DF^2 : DI^2$. Therefore $TF^2 : TI^2 :: DF^2 : DI^2$; and TF : TI : : DF : DI.

Cor. *If it be* TF : TI : : DF : DI, *the line* HK *joining the points of contact (of the tangents* TH, TK,*) will paſs thro' the point* D.

Fig. 38.

SCHOLIUM.

If one of the tangents be drawn to the oppoſite hyperbola, 'tis equally true.

PROP. XLVII.

39. *If* MT, AH *be two tangents at any points* M *and* A; *and if the ſemidiameters* CM, CA *be drawn, to interſect the tangents in* H *and* T. *Then the triangle* MNH = *triangle* ANT. *And the like for the oppoſite ſections.*

Draw PM parallel to AH. Then by ſimilar triangles, CM : CH : : CP : CA : : (XLII.) CA : CT. And alternately, CM . CA : : CH : CT; whence (Geom. II. 17. Cor. 1.) the triangle CAH = triangle CMT; which ſubtracted from the quadrilateral CANM, leaves MNH = ANT.

Cor. *The triangle* CAH = *triangle* CMT. From the demonſtration.

PROP. XLVIII.

40. *If* CM, CV *be any two conjugates. And* MP, QV *ordinates to any other diameter* AC; *and* CD *the conjugate of* AC; *then* $CQ^2 = CP^2 - CA^2 = APB$.

This Prop. is demonſtrated from Prop. XLII. and Cor. 1. Prop. XLIII. in the ſame manner as Prop. XXIX. was from Prop. XV. and Cor. 1. Prop. XXVIII. And the ſame corollaries will follow, viz.

Cor. 1. $CP^2 = CA^2 + CQ^2 = CA^2 + APB$.

Cor. 2. $CP^2 - CQ^2 = CA^2 = CP^2 - APB$.

Cor. 3. $QV^2 - PM^2 = CD^2$. Fig. 40.

Cor. 4. $CA : CD :: CP : QV :: CQ : PM$.

Cor. 5. *Hence the triangle* CPM = *triangle* CQV. This is proved the ſame way as in Prop. XXXVI.

PROP. XLIX.

If AH *be a tangent at any point* A; MC, VC, *two conjugate ſemidiameters, cutting the tangent in* Z *and* H; AC, CD *two conjugates. Then the rectangle* ZAH = CD^2. 41.

To the diameter AC draw the ordinates MP, VQ. Then by ſimilar triangles $CP : PM :: CA : AZ$, whence $CP \times AZ = CA \times PM =$ (Cor. 4. laſt Prop.) $CQ \times CD$; therefore $CP : CQ :: CD : AZ$.

Again, $CQ : QV :: CA : AH$, and $CQ \times AH = CA \times QV =$ (ib.) $CP \times CD$. Therefore $CP : CQ :: AH : CD :: CD : AZ$, and $AZ \times AH = CD^2$.

Cor. *If* SN *be drawn parallel to* CD, *then* $FN^2 - SFG = CD^2$.

For by ſimilar triangles $CA : AZ :: CF : FG$
and $CA : AH :: CF : FS$
and multiplying, $CA^2 : AZ \times AH$ or $CD^2 :: CF^2 : FS \times FG$.
and compounding $CA^2 : CD^2 :: CA^2 + CF^2 : CD^2 + SFG ::$ (XLIII.) $CA^2 + CF^2 : FN^2$; therefore $CD^2 + SFG = FN^2$, and $FN^2 - SFG = CD^2$.

PROP. L.

42. *If* PT, QT *be two tangents at* P, Q; *and* PQ *drawn between the points of contact*; *and if any line* LD *cuts* PQ, *and the curve in* S, A, B. *Then* SL^2 : SD^2 : : *rectangle* ALB : *rectangle* BDA.

Draw the ſemidiameters CG, CM, CF, parallel to TP, TQ, LD, (which I have put without the figure to avoid confuſion); and draw LI parallel to QT, to cut PQ in I. Then the triangles SLI, SQD, are ſimilar, as alſo PIL, PQT; whence SD : SL : : DQ : LI. But the ratio of DQ to LI is compounded of the ratio of DQ to PL, and (PL to LI or) PT to TQ. Therefore DQ : LI : : DQ × PT : : PL × TQ. Therefore SD : SL : : DQ × PT : PL × TQ. Take E a mean proportional between AL and LB; and H a mean proportional between BD and DA. Then ALB = E^2, and ADB = H^2.

Then (Prop. XL.) GC^2 : FC^2 : : LP^2 : ALB or E^2, and FC^2 : MC^2 : : BDA or H^2 : DQ^2. Therefore extracting the roots, GC : FC : : LP : E; and FC : MC : : H : DQ; therefore GC : MC : : PL × H : DQ × E. But (XL. Cor. 1.) PT : QT : : GC : MC : : PL × H : DQ × E. And multiplying means and extremes, PT × DQ × E = QT × PL × H. Whence H : E : : DQ × PT : PL × TQ : : SD : SL; and H^2 or ADB : E^2 or ALB : : SD^2 : SL^2.

Cor. *Hence in any inſcribed trapezium* LNDO, *the diagonals* LD, NO, *interſect in the ſame point* S, *where the lines* PQ, KR *interſect, that join the oppoſite points of contact.*

For if LD cuts the tangents PL, QD in L and D, and PQ in S; we have, SL^2 : SD^2 : : ALB : BDA : :

BDA : : EE : HH. And SL : SD : : E : H; and compounding, SL + SD : SL : : E + H : E; or E + H : E : LD : SL. Fig. 42.

Alſo for the tangents KL, RD, it will be SL^2 : SD^2 : : ALB : ADB. Whence, as before, it will be E + H : E : : LD : LS. Therefore PQ, and KR interſect LD in the ſame point S. And by the ſame reaſoning, the diagonal NO, cutting the conjugate hyperbolas in *a* and *b*, will paſs thro' S, the interſection of PQ, KR.

PROP. LI.

43. *If* MG *be any diameter*, MD, GT *two tangents at* M *and* G; DTN *a tangent at* N, *cutting the former in* D *and* T. *And if the ſemidiameter* CK *be drawn parallel to* ND. *Then the rectangle* DNT = CK^2.

For let CF be the ſemiconjugate of MG, then (XL. Cor. 1.)

CF : CK : : DM : DN.
and CF : CK : : TG : TN.
multiplying CF^2 : CK^2 ; DM × TG : DN × TN.
But (XLIV.) CF^2 = DM × TG; therefore CK^2 = DN × NT.

PROP. LII.

44. *If* FG *be drawn thro' the points of contact of two tangents* AF, AG; *and* AV *be drawn thro'* A *their interſection, parallel to* GF. *And if from any point in it as* V, LVO *be drawn thro' the middle of* FG, *to cut the curve in* P *and* L; *then will* VP : VL : : OP : OL.

From L draw LAQR, cutting the curve in Q, from Q draw QSP parallel to GF. Then (Prop.

Fig. XLVI.) LA : LR : : QA : QR, and LA × QR
44. = LR × QA.

The ratio of PQ to SQ is compounded of the ratio of PQ to OR and OR to SQ. But by ſimilar triangles PQ : OR : : LQ : LR; and OR : SQ : : AR : AQ. Therefore PQ : SQ : : LQ × AR : LR × AQ. And by diviſion, PS : SQ : : LQ × AR — LR × AQ : LR × AQ. But LQ × AR — LR × AQ = $\overline{\text{LR} - \text{QR}}$ × AR — LR × AQ = LR × $\overline{\text{AR} - \text{AQ}}$ — QR × AR = LR × QR — AR × QR = AL × QR = LR × AQ before. Therefore PS : SQ : : LR × AQ : LR × AQ; and therefore PS = SQ, and conſequently, ſince QS is an ordinate, PS is an ordinate, and P is in the curve. Laſtly, by reaſon of the parallels FG, PQ, VA, VP : VL : : AQ : AL : : (before) RQ : RL : : OP : OL.

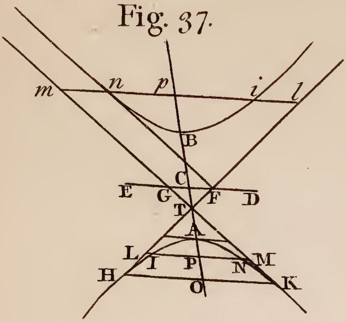
PROP. LIII.

45. *If* FG *join the points of contact of two tangents* FA, GA; *and* AV *be parallel to* GO. *Then if two tangents* VH, VI, *be drawn from any point* V *in it. Then* HI *drawn thro' the points of contact, will paſs thro' O the middle of* FG.

For let B, L be two points in the curve; where VO, AO cut it. Then in reſpect of the ordinate FG, it is (by laſt Prop.) VP : VL : : OP : OL. And in reſpect of the line HI, it is (Prop. XLVI.) VP : VL : : OP : OL; therefore O is common to both lines FG, HI; and therefore HI paſſes thro' O.

Cor. 1. *If* FA, GA *be two tangents, and* FG *drawn thro' the points of contact. Alſo* HV, IV *two other tangents, and* HI *drawn thro' their points of contact. Then if* AV *be drawn thro' the interſections of*

Fig. 37.



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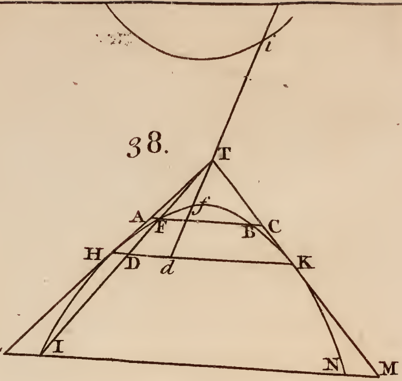

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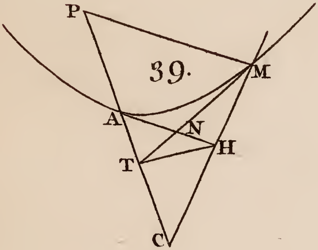

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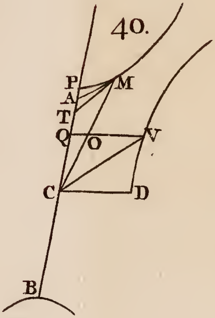

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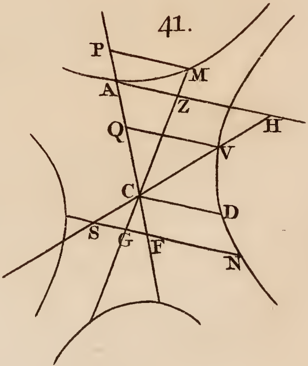

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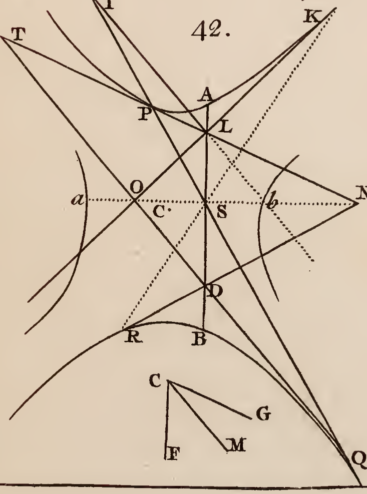

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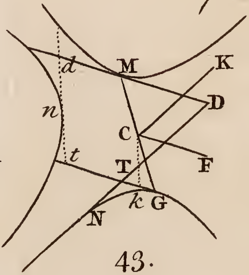

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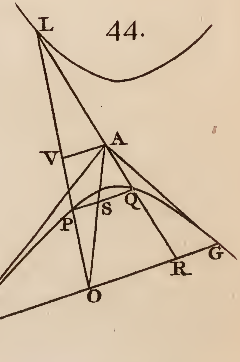


Pl. XVII. *pa. 120.*

of the tangents; then will O, *where* FG, HI *interſect, be the point thro' which all lines paſs, which join the points of contact of any two tangents, which meet in the line* AV. Fig. 45.

Cor. 2. *If* FA, GA *be two tangents, and if* AV *be drawn parallel to* FG; *then if any line* HI *be drawn thro' the middle of* FG; *then the tangents drawn from* H *and* I *will meet ſome where in the line* AV.

Otherwiſe it could not be VP : VL : : OP : OL, as by Prop. LII.

Cor. 3. *If* IH *be continued to interſect* AV *in* D, *and the tangents* DP, DL *drawn; then* LP *paſſing thro' the points of contact will paſs thro'* O.

PROP. LIV.

If VH, VI *be two tangents, and* IHD *be drawn thro' the points of contact; and if any line* OPVL *be drawn thro'* V, *to cut the curve in* P *and* L; *then two tangents* PD, LD *drawn from* P *and* L, *will meet ſome where in the line* ID. 45.

Let O be the interſection of VP and IH; thro' O draw FG, ſo that FO = OG, and draw VAD parallel to it. Then (LIII. Cor. 2.), the tangents PD, LD, will meet in the line VA, as at D. Then if DO does not paſs thro' the points of contact H, I; let a line drawn from D to N paſs thro' the points of contact. Then (Prop. XLVI.) we ſhall have VP : VL : : NP : NL. But HV, IV, being tangents, it will be (by the ſame Prop.) VP : VL : : OP : OL. Therefore the former proportion is falſe, except N coincide with O. Therefore DO paſſes thro' the points of contact H, I. Or which is

Fig. is the ſame, the tangents PD, LD meet in the line IH.

46. Cor. *If* FA, GA *be two tangents, and* GBF *be drawn thro' the points of contact. And from any point* B *in* FG, *the tangents* BI, BH *are drawn. Then* IH *(drawn thro' the points of contact* H, I.*) will paſs thro'* A.

For a line drawn from A will paſs thro' the points of contact H, I; and therefore no other line can paſs thro' them.

PROP. LV.

47. *If a line* MN *be drawn thro' the focus* F, *and tangents* MR, NR *drawn from* M *and* N. *Then the line* RF *drawn from the interſection of the tangents to the focus, will be perp. to the line* MN.

Let C be the center, and make CF, CA, CT continually proportional. At T erect TL perp. to the tranſverſe axis BA. Then (Prop. XV.) a tangent drawn from the end of the ordinate erected at F, will cut the axis at T. And (LIII. Cor. 2.) The tangents drawn from M and N, will meet ſome where in the line TL, as at R. Make SG = BA, draw GF, which (Prop. XII.) will be perp. to MR at D. Draw MLO parallel to BA.

The triangles SFG, MOG are ſimilar, whence SG or BA : SF : : MG : MO; that is, CA : CF : : MF : MO, But (Prop. XXVI.) CF : CA : : ML : MF. Therefore ML : MF : : MF : MO, and $ML \times MO = MF^2$. But the triangles MDO, MLR are ſimilar, being right angled at D and L. Whence MD : MO : : ML : MR; therefore $MD \times MR = ML \times MO = MF^2$. And MD : MF : : MF : MR. Whence the triangles MDF, MFR, are ſimilar, and therefore the angle MFR = ∠ MDF = a right angle.

Cor. *A circle deſcribed about the diameter* MR *will* Fig.
paſs thro' the focus F, *and thro' the points* L, G. 47.
For MFR and MLR are right angles, and the triangle MGD being equal and ſimilar to MFD, will be contained in the oppoſite ſemicircle.

PROP. LVI.

If two tangents HE, HK *touching the conjugate hyperbolas in* R *and* D, *cut any diameter* AB *in* E *and* K, *ſo that* AE = BK. *And if* FZ *be a third tangent; touching the curve at* I, *and cutting the former in* F *and* G. *Then the rectangle* EF × GK = KH × RE, *a given quantity.* 48.

For ſince CERA is ſimilar and equal to CPKB; therefore RE = PK. And (Prop. XLIV. Cor. 3.) PK : RF : : KG : GH, and compounding PK or RE : RE + RF : : KG : KG + GH; that is, RE : EF : : KG : KH; whence EF × KG = KH × RE a given quantity.

Cor. *Producing* LE *and* ZF *to* V; *then* VE × KZ = KH × RE, *a given quantity.*
For LZ is parallel to EH, and VL to HK; therefore the triangles VEF and GKZ are ſimilar; and VE : EF : : GK : KZ, whence VE × KZ = EF × GK = KH × RE.

PROP. LVII.

If ABFE *be an inſcribed trapezium; and if the diagonals* AF, BE *be biſſected. The line drawn thro' the points of biſſection will paſs thro' the center* C. 49.

Thro' the points of contact of the oppoſite ſides AE, BF, draw the line HI, and the diameter GD parallel to it. Then (Prop. LVI.) GA × DB = a given quantity = GE × DF. Whence GA : GE : :

DF :

Fig. 49. DF : DB. Therefore if AF, GD, EB be drawn, then (Cor. 1. Lemma to Prop. LX. Ellipſis), the line that paſſes thro' the middle of AF, EB, paſſes thro' the middle of GD. But as GD is parallel to HI, the middle of GD is the center C.

Cor. *And if there be any four tangents to the hyperbola, and lines (or diagonals) be drawn to the oppoſite interſections. Then a line drawn thro' the middle of theſe, will paſs thro' the center.*

This is ſhewn the ſame way as was done in the ellipſis.

PROP. LVIII.

50. *If* AB *be any diameter; and if two lines* AP, BP *be drawn thro' any point in the curve as* P; *and the ordinate* DF *be drawn interſecting the lines in* E *and* G. *Then* DE, DF, DG *are in continual proportion.*

Draw the tangents AN, BH, the conjugate ICL, and ordinate PO. Then by ſimilar triangles

AB : BH : : AO : OP
and AB : AN : : BO : OP

multiplying, AB^2 : AN × BH : : AO × OB : OP^2 : : (XXXVI.) AB^2 : IL^2; therefore AN × BH = IL^2.

Again (XXXVI.) AB^2 : ADB : : IL^2 : DF^2; and by ſimilar triangles

AB : AD : : BH : DE
and AB : BD : : AN : DG

multiplying, AB^2 : ADB : : AN × BH : DG × DE.
therefore IL^2 : DF^2 : : AN × BH : DG × DE.
But IL^2 = AN × BH; therefore DF^2 = DG × DE, or DE, DF, DG ∺.

Cor. *If from the ends of any diameter* AB, *two lines* PH, PB *be drawn thro' any point* P *of the curve,*

curve, to interſect the tangents BH, AN *in* H *and* Fig.
N. *The rectangle of the tangents is equal to the* 50.
ſquare of the conjugate. $AN \times BH = IL^2$.

This appears from the demonſtration.

PROP. LIX.

If C *be the center,* F *the focus, and if* CF : CA : : CA : CT, *and the directrix* TO *be drawn perp. to* AB. *And if any line* MNO *be drawn, and the lines* FN, FO, MFH *be drawn to the focus. Then* OF *biſſects the angle* NFH. 51.

Draw MP, NQ perp. to TO, and ND parallel to MF, cutting FO in D. Then (by ſimilar triangles) MP : NQ : : MO : NO : : MF : ND; and alternately, MP : MF : : NQ : ND. But (XXVI.) MP : MF : : CA : CF : : NQ : NF; therefore NQ : ND : : NQ : NF. And ND = NF. Therefore $\angle$ NFD = NDF = (by the parallels) DFH.

PROP. LX.

If AB *be any diameter,* BG *a tangent at* B *equal to the latus rectum. If* AG *be drawn cutting the ordinate* PM *(produced) in* D. *Then* $PM^2 = BP \times PD$. 52.

For (XXXVI. Cor. 1.) AB : BG : : APB : PM^2. And by ſimilar triangles, AB : BG : : AP : PD : : $AP \times PB$: $PD \times PB$: : APB : PM^2. Therefore $PD \times PB = PM^2$.

Cor. *Hence* PM^2 *the ſquare of the ordinate, is always greater than* $PB \times BG$, *the rectangle of the parameter and abſciſſa; by the rectangle* $GF \times FD$, *which is ſimilar to the rectangle* $AB \times BG$. *And by reaſon of that exceſs the curve is called an HYPERBOLA.*

For

Fig. 52. For if GF be parallel to AB, the rectangle DFG is ſimilar to ABG, for by the ſimilar triangles ABG, GFD, AB : BG : : GF : FD.

SCHOLIUM.

The rectangle under the diameter AB, and its parameter BG, is called the *Figure* of that diameter.

PROP. LXI.

53. *If* QT, PT *be two tangents, and if* QF, PF *be drawn from the points of contact to the focus* F; *then the line* TF *will biſſect the angle* QFP.

Make CF : CA : : CA : CG, and draw the directrix DGR perp. to AB. Produce FT to K, and draw LR, KR (Prop. L.) tangents at L, K. Let QD, PE be perp. to DR. Then (XXVI. Cor.) QF : PF : : QD : PE : : (ſimilar triangles) QR : PR : : (XLVI.) QN : PN; therefore (Geom. II. 25. Cor. 1.) FN biſſects the angle PFQ.

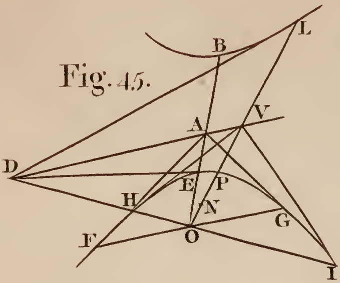
PROP. LXII.

54. *If* CB *be the ſemitranſverſe,* BD *the ſemiconjugate axis*; S, F *the foci,* PM *an ordinate. Thro'* C *and* D, *draw* CO, *to cut the ordinate* PM *in* O, *and make* $CI = CO$. *Then* $SM = BI$.

For (Cor. 1. Prop. V.) CB : CS : : CP : SM + CB. Whence $SM \times CB + CB^2 = CS \times CP$; and $SM \times CB = CS \times CP - CB^2$. But (def. 6.) $CD = CS$, and by ſimilar triangles, CB : CD or CS : : CP : CO or CI, and $CS \times CP = CB \times CI$; therefore $SM \times CB = CB \times CI - CB^2$; whence by diviſion $SM = CI - CB = BI$.

Cor. 1. *The ſame things remaining*, $FM = AI$.

Fig. 45.



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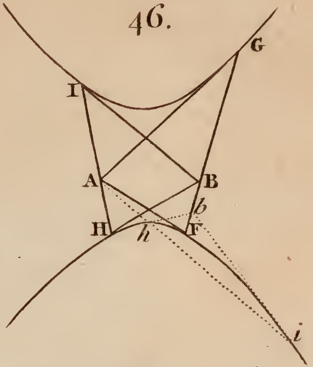

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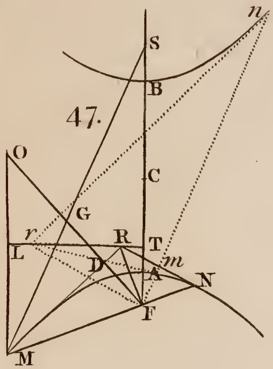

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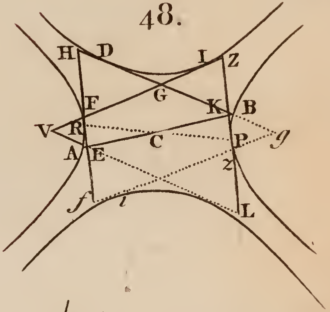

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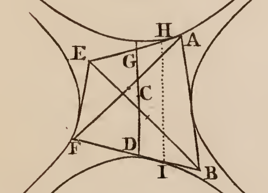

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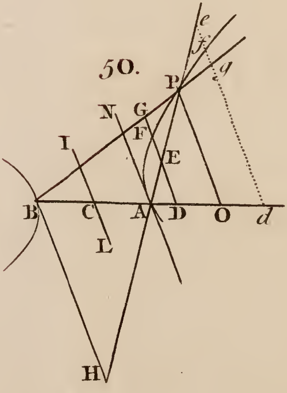

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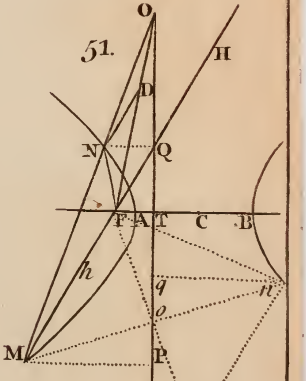


Pl. XVIII. *pa.* 126.

For $FM = SM + AB = BI + BA = AI$. Fig. 54.

Cor. 2. *If* MS *be produced to* N, *ſo that* $SN = SC$, *and* NL *be perp. to* CB. *Then* $CB - SL$, BD, *and* SM *are continually proportional; but when* P *is between* S *and* B, *it is* $CB + SL$.

For by ſimilar triangles, $SM : SP : : SN$ or $SC : SL$; whence $SL \times SM = SC \times SP = CF \times SP = CF \times \overline{CP - CS} = CF \times CP - CS^2$. But we had $CF \times CP$ or $CS \times CP = SM \times CB + CB^2$; therefore $SL \times SM = SM \times CB + CB^2 - CS^2$; and $SM \times \overline{CB - SL} = CS^2 - CB^2 =$ (II. Cor.) BD^2. But if P is between S and B, $SL \times SM = CF \times \overline{CS - CP}$, &c.

Cor. 3. *Put* Z *for the coſine of the angle* BSM, *then will* $SM = \frac{CD^2}{CB + Z \times CS}$. *Or when* BSM *is greater than a right angle, it is* $CB - Z \times CS$.

For by trigonometry, Rad. (1.) : SN or SC : : S.SNL (Z) : $SL = Z \times SC$, and $CB + SL = CB + Z \times SC$.

PROP. LXIII.

55. *If* ABDC *be an inſcribed trapezium, and any point* E *be taken in the curve, from which the two lines* EN, EH *are drawn parallel to any two adjoining ſides as* AB, AC, *to interſect the oppoſite ſides in* N *and* Q, *and in* H *and* R. *Then taking the rectangles upon the oppoſite ſides;* $ER \times EH$ *will always be to* $EN \times EQ$, *in a given ratio, wherever* E *is placed.*

Draw DFG, BPL parallel to AC; thro' C and L draw COLS; and thro' the middle of AC, BL, draw the diameter MK, which will biſſect the parallels DF, TE; and alſo SG, OR which are terminated

Fig. 55. minated at AB, CL. Then will SD = FG, OT = ER; and theſe would alſo be equal if SG, RH were tangents.

By reaſon of the parallels, and from the ſimilar triangles, HO : SD : : OC : SC : : AR : AG; and alternately, HO : AR or EQ : : SD or FG : AG.

Again, BP or ER : PN : : DG : GB; and multiplying the reſpective terms, HO × ER : EQ × PN : : FG × DG : AG × GB : : (XXXIX. Cor. 2.) ER × RT or ER × EO : RA × RB or EQ × EP. And compounding, FG × DG : AG × GB : : HO × ER + EO × ER : EQ × PN + EQ × EP : : EH × ER : EQ × EN. But it is evident, as long as the points A, B, D, C, are fixed; that FGD to AGB is a given ratio, and conſequently that of REH to QEN.

Cor. 1. *The ſame things ſuppoſed, it will be* EH : EN : : EO : EP : : HO : PN.

For EH × ER : EQ × EN : : FG × GD : AG × GD : : HO × ER : EQ × PN. And alternately, EH × ER : HO × ER : : EQ × EN : EQ × PN; and by dividing equally, EH : HO : : EN : PN; and by diviſion, EH : EO : : EN : EP.

Cor. 2. *If the points* A, B, C, E, *be fixed; and* D *any way changed; it will ſtill be* EH : EN : : HO : PN : : EO : EP, or RT : RB *a given ratio.*

Cor. 3. *If the points* A, C, *coincide*; MQ *becomes a tangent. And if* B, D, *coincide*; BN *becomes a tangent.*

56. Cor. 4. *If* BC *be drawn to cut* EH *in* I; *and if it be made as* EH : EI : : EN : EV. *Then* BV *being drawn will touch the curve in* B.

For (Cor. 2.) when D is moveable, it is, EH : EN : : EO : EP. And if D comes to B, then

(Cor. 3.) BND becomes a tangent; in which caſe Fig.
H comes to I and N to V; and then it is, EH : 56.
EN : : EO : EP : : EI : EV.

Cor. 5. *If we make* EH : EN : : EO : EP, *and* 55.
draw CO, BP; *they will meet at* L *in the curve.*
By Cor. 2.

Cor. 6. *If there be five points*, B, L, D, C, E; *and thro' any point* E, *there be drawn* EH, EN, *cutting* BD, BL, DC, LC, *in* N, P, H *and* O; *ſo that* EH : EN : : EO : EP. *Then* CA, BA *being drawn parallel to* EH, EN, *they will meet at* A *in the curve.*

This is the reverſe of the foregoing. For if A was out of the curve, L would not be in it.

PROP. LXIV.

If ABDC *be an inſcribed trapezium, and from* 57.
any point E, *of the curve, four lines* EL, EK, EO, EM, *be drawn to the four ſides of the trapezium, in any angles whatever, equal or unequal; cutting them in* L, K, O, M. *And likewiſe from any other point e, four more lines, el, ek, eo, em, be likewiſe drawn reſpectively parallel to the former, cutting the ſides in l, k, o, m.*

Then taking the rectangles of the lines falling upon the oppoſite ſides, it will be, rectangle EL × EM : *el* × *em* : : EK × EO : *ek* × *eo.*

Thro' E, *e*, draw EQN, *eqn*, parallel to AB; and ERH, *erh*, parallel to AC. Then by ſimilar triangles, ER : *er* : : EL : *el*; and EH : *eh* : : EM : *em*. And multiplying, ER × EH : *er* × *eh* : : EL × EM : *el* × *em*.

Again, EQ : *eq* : : EK : *ek*, and EN : *en* : : EO : *eo*. And multiplying, EQ × EN : *eq* × *en* : : EK × EO : *ek* × *eo*. But (Prop. LXIII.) ER × EH : *er* ×

Fig. 57. *er* × *eh* : : EQ × EN : *eq* × *en*. Therefore EL × EM : *el* × *em* : : EQ × EN : *eq* × *eh* : :) EK × EO : *ek* × *eo*.

58. Cor. 1. *In the inſcribed trapezium* ABCD; *if* AB, CD, *produced, interſect in* X; *and the diagonals* AC, BD, *in* Y; *and* YEX*e be drawn. Then* EX : *e*X : : EY : *e*Y.

For ſuppoſe BD, AC to become the diagonals; and that EL, EK, EO, EM, and *el*, *ek*, *eo*, *em*, all lie in one line E*e*; and that AB, CD, interſect in X, and AC, BD, in Y; and that E*e* paſſes thro' X, Y; which is a particular caſe of this Prop. Then L, *l*, M, *m*, coincide in X; and K, *k*, O, *o*, in Y. Then the proportion EL × EM : *el* × *em* : : EK × EO : *ek* × *eo*, becomes EX^2 : eX^2 : : EY^2 : eY^2, and EX : *e*X : : EY : *e*Y.

Cor. 2. *If* XA, XC, *be two tangents, and* AYC *paſſes thro' the points of contact. It will be* EX : *e*X : : EY : *e*Y.

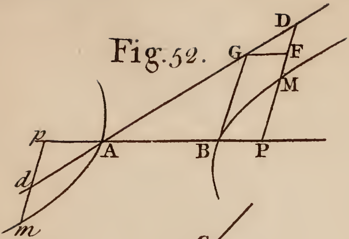
For let B approach to A, and D to C; then BD will coincide with AC. And it will ſtill be EX : *e*X : : EY : *e*Y, as in Prop. XLVI.

PROP. LXV.

59. *If any line* MN *be drawn thro' the focus* F, *and* ZCQ *be a diameter drawn parallel to it*; AB *the tranſverſe axis. Then* AB, QZ, MN, *are continually proportional.*

Draw the diameter CDL thro' the middle MN, and the ordinate MP perp. to AB, and tangent MTO. Then (VII. Cor. 3.) CF × CP = CA × FM + CA^2, and CA × FM = CF × CP — CA^2 = (XV.) CF × CP — CT × CP, or CA × FM = TF × CP. And by ſimilar triangles CT : CO : : TF :

Fig. 52.



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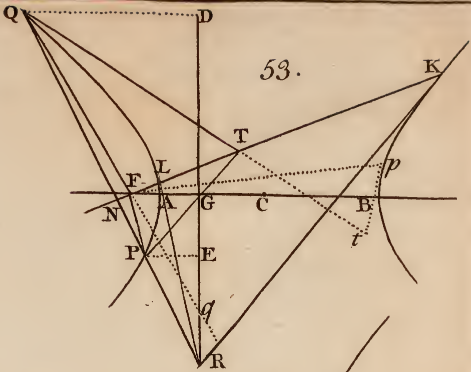

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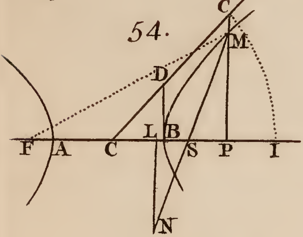

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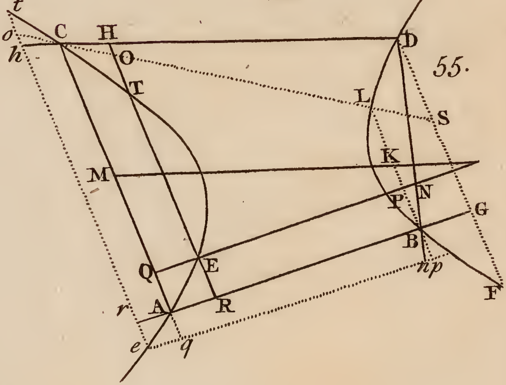

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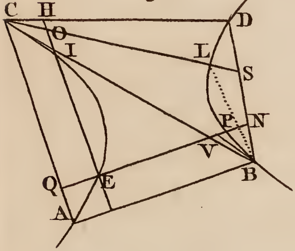

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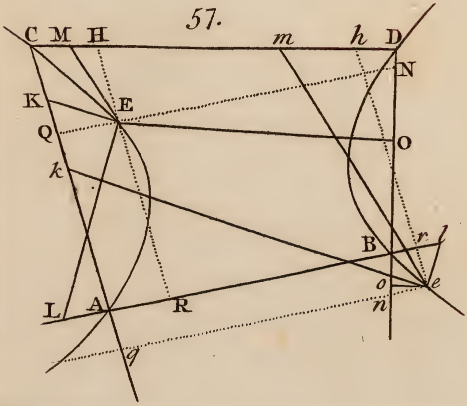

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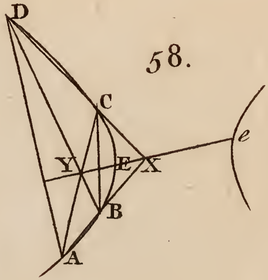


Fig. 59. $TF : FM :: TF \times CP$ or $CA \times FM : FM \times CP :: CA : CP :: CT : CA$. Whence $CO = CA$.

But (XLII. Cor. 5.) $CZ^2 = CO \times CG = CA \times LM$; and multiplying, $QZ^2 = BA \times MN$.

Cor. 1. *If* F *be the focus*, ZQ *a diameter parallel to* MF; MT *a tangent at* M, *cutting* ZQ *in* O; *then* $CO = CA$.

From the demonſtration.

Cor. 2. *If* AB *be the axis*, F *the focus*, MP *an ordinate*, MT *a tangent. Then* $CA \times FM = TF \times CP$.

From the demonſtration.

Cor. 3. *If any line* MN *be drawn thro' the focus* F; *the rectangle* $MFN = \frac{1}{2} MN \times \frac{1}{2}$ *latus rectum.*

Let $L = \frac{1}{2}$ latus rectum; then (Prop. XXXVIII.) $CZ^2 : MFN :: CA^2 : AFB ::$ (IV. Cor. 3.) $CA^2 : CA \times \frac{1}{2} L :: CA : \frac{1}{2} L :: CA \times ML : \frac{1}{2} L \times ML$. But $CZ^2 = CA \times ML$; therefore $MFN = \frac{1}{2} L \times ML$.

PROP. LXVI.

60. *In an equilateral hyperbola, the ſquare of the ſemitranſverſe is equal to the rectangle of the diſtances of either vertex from the foci.* $CA^2 = SAF$.

For (Prop. III.) the rectangle of the diſtances of the two foci, from either vertex, is equal to the ſquare of the ſemiconjugate. But (Def. XII.) the tranſverſe and conjugate are equal; therefore that rectangle is equal to the ſquare of the ſemitranſverſe axis.

PROP. LXVII.

60. *In an equilateral hyperbola, the rectangle of the diſtances of the ordinate from the two vertices, is equal to the ſquare of the ordinate.* $APB = PM^2$.

For (Prop. VI.) as ſquare of the tranſverſe : ſquare of the conjugate : : ſo the rectangle of the diſtances of the ordinate, from the two vertices BPA : to ſquare of the ordinate. But the two firſt terms are equal (Def. XII.). Therefore the two latter are equal.

Cor. $CP^2 - CA^2 = PM^2$, C *being the center.*

PROP. LXVIII.

61. *If* AV *be an equilateral hyperbola,* AC *the ſemitranſverſe axis,* CQ *perp. to* CA; QV *parallel to* CA. *Then* $CA^2 + CQ^2 = QV^2$.

For (Prop. XXVIII.) if CD be the ſemiconjugate, then $CA^2 : CD^2 :: CA^2 + CQ^2 : QV^2$. But $CA^2 = CD^2$; therefore $CA^2 + CQ^2 = QV^2$.

Cor. *The hypothenuſe* AQ = *ordinate* QV *parallel to* CA.

For $AQ^2 = CA^2 + CQ^2 = QV^2$; and $AQ = QV$.

PROP. LXIX.

62. *If* CA *be the ſemitranſverſe axis,* CF, CH, *the aſſymptotes;* AD *the ſemiconjugate;* PM *an ordinate continued to* F. *The rectangle* $HMF = AD^2$.

By ſimilar triangles, $CP^2 : PF^2 :: CA^2 : AD^2 ::$ (VI.) APB or $CP^2 - CA^2 : PM^2 ::$ (by diviſion) $CP^2 - CP^2 + CA^2$ or $CA^2 : PF^2 - PM^2$; that

is, $CA^2 : AD^2 :: CA^2 : PF^2 - PM^2$; therefore Fig.
$AD^2 = PF^2 - PM^2 = HMF$. 62.

Cor. 1. *If the ordinate be continued to the aſſymptotes, the parts between the curve and aſſymptotes are equal.* $FM = GH$.

For $PH = PF$, and $PG = PM$; therefore the remainders are equal, $FM = GH$.

Cor. 2. *If more ordinates be drawn to cut the aſſymptotes, their rectangles are all equal.*

PROP. LXX.

If from the point M *of the curve,* MK *be drawn* 62.
parallel to the tranſverſe axis AB, *cutting the aſſymptotes in* I *and* K; *then the rectangle* $KMI = CA^2$.

For the triangles IMF, KMH and CAD are ſimilar, whence $AD : AC :: MF : IM$, and $AD : AC :: MH : KM$; and multiplying, $AD^2 : AC^2 :: MF \times MH : IM \times KM$. But (LXIX.) $MF \times MH = AD^2$; therefore $IM \times KM = AC^2$.

Cor. 1. *The rectangle* $IM \times MH = KM \times MF$. For by ſimilar triangles $KM : MH :: IM : MF$.

Cor. 2. *If more lines be drawn parallel to* CA; *all their rectangles are equal to one another.*

PROP. LXXI.

If two parallel lines PS, FH, *cut the hyperbola,* 63.
and the aſſymptotes; the rectangle FMH = *rectangle* 64.
PQS.

Thro' M and Q draw OL, IN perp. to the axis. Then by reaſon of the parallels, the triangles PQI, FMO, are ſimilar, as alſo MHL, QSN. Therefore $MO : MF :: QI : QP$, and $ML : MH ::$

Fig. $QN : QS$. And multiplying $MO \times ML : MF \times$
63. $MH :: QI \times QN : QP \times QS$. But (LXIX.
64. Cor. 2.) $MO \times ML = QI \times QN$; therefore $FMH = PQS$.

Cor. 1. *If the tangent* DG *be parallel to* FH, *then the rectangle* $FMH = DG^2$.

Cor. 2. *If the diameter* AB *be parallel to* HM; *then rectangle* $FMH = CB^2$.

PROP. LXXII.

65. *Any right line* FMGH *be drawn thro' the curve; the parts of the line between the curve and aſſymptotes, are equal*; $FM = GH$.

For (Prop. LXXI.) rectangle $FMH = HGF$; that is, $FM \times MG + FM \times GH = GH \times GM + GH \times MF$; and $FM \times MG = GH \times GM$; whence $FM = GH$.

Cor. 1. *Hence* $FG = MH$.

Cor. 2. *If* AB *be a tangent at* D; *then* $AD = DB$, *or the tangent is biſſected in* D.
For in this caſe M and G coincide in D.

Cor. 3. *If a diameter* CD *be drawn thro' the point of contact* D, *it biſſects all the parallels* MG, *and all the lines* FH.

PROP. LXXIII.

66. *If from two points in the curve*, M, Q, *there be drawn the parallels* MH, QN, *to cut one aſſymptote; and likewiſe the parallels* MO, QI, *to cut the other aſſymptote. Then the rectangle* $MO \times MH =$ *the rectangle* $QI \times QS$.

Draw EML, PQN perp. to the axis. Then the triangles EMO, PQI are ſimilar; and alſo

MLH, QNS. Whence $ME : MO :: QP : QI$; Fig.
and $ML : MH :: QN : QS$. And multiplying, 66.
$ME \times ML : MO \times MH : QP \times QN : QI \times QS$.
But (LXIX. Cor. 2.) $ME \times ML = QP \times QN$; therefore $MO \times MH = QI \times QS$.

SCHOLIUM.

The Prop. holds equally true, if QS, &c. cut the aſſymptote beyond C towards *h*.

PROP. LXXIV.

If two lines KN, FT *cut one another in* E; *and* 67.
likewiſe cut the curve and aſſymptotes; then rectan- 68.
gle $LKM : DFV :: KEN : FET$.

Thro' N, draw IG parallel to VD; then the triangles LNG, and LED are ſimilar; and alſo MNI, MEV; whence $LE : LN :: ED : NG$, and $ME : MN :: EV : NI$. And multiplying, $LE \times ME : LN \times MN :: ED \times EV : NG \times NI$.

But $LE \times ME = \overline{LK + KE} \times \overline{EN + NM} = LK \times EN + KE \times EN + \overline{LK + KE} \times NM = KE \times EN + LK \times EM + KE \times NM\ (KE \times LK) = KE \times EN + LK \times KM$ (fig. 67.).

Alſo by the ſame reaſoning, $DE \times VE = FET + DFV$. Alſo (LXXII.) $LNM = LKM$, and $GNI = DFV$. Whence we have
$KEN + LKM : LKM :: FET + DFV : DFV$.
And by diviſion, $KEN : LKM :: FET : DFV$.

PROP. LXXV.

All inſcribed parallelograms, between the aſſymp- 69.
totes and the curve, are equal; $ABCD = GFCH$.

For (by Prop. LXXIII.) the rectangle $AB \times AD = GF \times GH$; whence $AB : FG :: GH :$

Fig. 69. AD, or CD : CH : : CF : CB; therefore (Geom. III. 9. Cor. 1.) parallelogram AC = parallelogram GC.

Cor. 1. *The hyperbola continually approaches to the aſſymptote; but can never touch it, till it be infinitely continued.*

For EL being reciprocally as CE, EL continually diminiſhes as CE increaſes, and when CE is infinitely great, EL will be infinitely ſmall.

Cor. 2. *If a line be drawn parallel to one aſſymptote, it will interſect the oppoſite hyperbolas in one point, and no more.*

For AD can only cut the curve in A, for the curve AL afterwards approaching CE, muſt diverge from AD. And the ſame line continued can never cut the oppoſite hyperbola, becauſe it lies wholly without the oppoſite angle, in which the curve is wholly contained.

Cor. 3. *The aſſymptote may be conſidered as a tangent to the curve at an infinite diſtance.*

From Cor. 1. and Cor. 4. Prop. XV.

PROP. LXXVI.

70. *If the lines* AB, DF, *touch the hyperbola; then the triangles* CAB *and* CDF *made by the tangents, and aſſymptotes, are equal.*

From the points of contact, E, G, draw EH, GI parallel to the aſſymptote CB; and EO, GP, parallel to the aſſymptote CF. Then (LXXII. Cor. 2.) ſince AE = EB, and FG = GD; therefore AH = HC, and CO = OB; alſo FI = IC, and CP = CD.

But triangle ACB : to triangle FCD : : CA × CB or 2CH × 2CO : FC × CD or 2CI × 2CP : : CH ×

CH × CO : CI × CP : : parallelogram CHEO : parallelogram CIGP; but (Prop. LXXV.) CHEO = CIGP; therefore triangle ACB = FCD. Fig. 70.

Cor. 1. *All inſcribed triangles are equal to one another, and double to the inſcribed parallelograms.*

For triangle ACB : parallelogram CHOE : : $\frac{2CH \times 2CO}{2}$ or 2CH × CO : CH × CO : : 2 : 1.

Cor. 2. *If* GI *be parallel to the aſſymptote* CB; *and* GF *a tangent at* G; *then the ſubtangent* IF = CI *the diſtance from the center.*

For DF is biſſected in G; and therefore CF is biſſected in I.

Cor. 3. *Any two tangents to the curve* LG *interſect one another within the angle of the aſſymptotes* BCF.

For if BA be a tangent at E, this interſects CF (the tangent at an infinite diſtance) in A. And therefore the ſame BA interſects any other tangent FD, between A and B.

Cor. 4. *Hence alſo the rectangle* ACB *is equal to the rectangle* FCD.

Cor. 5. *If* AD *and* FB *be drawn, they will be parallel.*

For CA : CD : : CF : CB, from the equal rectangles CAB, CDF.

Fig.

PROP. LXXVII.

71. *If thro' any point* F *in the curve, a line* FP *be drawn cutting the curve and aſſymptotes in* P, B, M; *and from* B *the tangent* BEA; *and from* E *the point of contact*, EH; *as likewiſe* FQ, PR *all parallel to* BC. *Then* $2EH = FQ + PR = FQ - pr$ *in the oppoſite ſection.*

For draw FL parallel to CA; then the triangles MPR, FBL, are ſimilar, and $MP = PF$; therefore $PR = LB$, and $LC = FQ$; and ſince $AE = EB$ (Prop. LXXII. Cor. 2.) $= \frac{1}{2} AB$; therefore $HE = \frac{1}{2} CB$, and $2HE = CB = CL + LB = QF + PR$; or $2he = QF - pr$.

Cor. 1. *Hence the ſum of any two lines* FQ, PR *(made by drawing any line* BP, *from the ſame point* B *in the aſſymptote), will always be the ſame.*

For that ſum is always equal to 2EH.

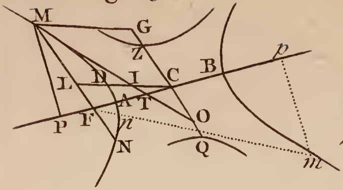
Cor. 2. *If* BP *be parallel* CM; *then* $2EH = FQ$.

PROP. LXXVIII.

72. *If* AC *be the ſemitranſverſe of the equilateral hyperbola* AM, CF *its aſſymptote, and* CH *perp. to* CA, *and* HM *parallel to* CA. *Then if* MF *be drawn perp. to the aſſymptote*, $2CH \times HM = CF^2 - MF^2$.

For draw the ſemiconjugate AD, which is equal to AC. The triangles CAD, CHV, VFM are ſimilar, and iſoceles; whence $CH = HV$, $VF = MF$. And $VF : VM :: VH : CV$, and $VM \times VH = VF \times CV$. Then $CV^2 = 2HV^2$, and $2CH \times HM = 2HV \times HM = 2HV \times \overline{HV + VM}$

Fig. 59.



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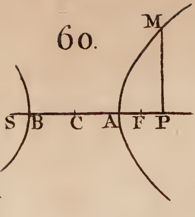

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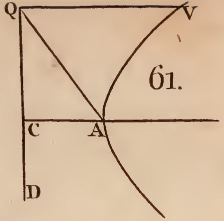

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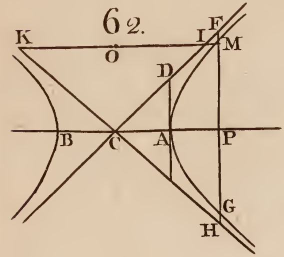

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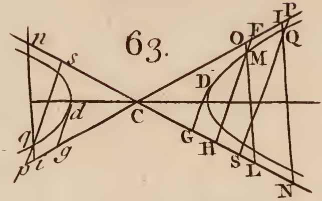

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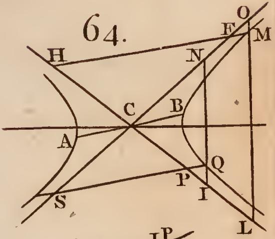

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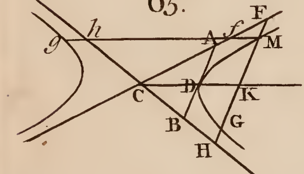

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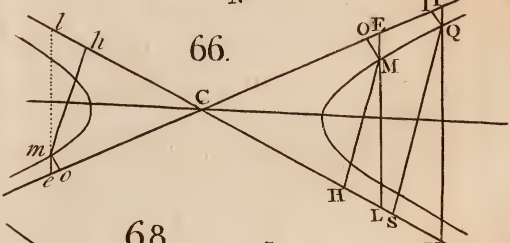

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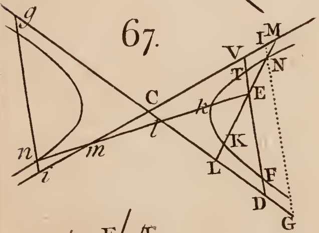

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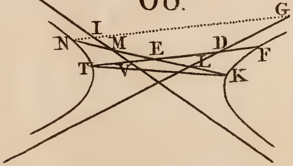

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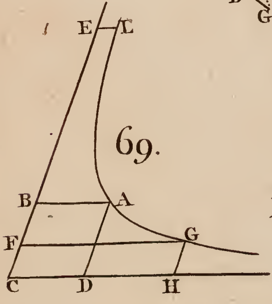

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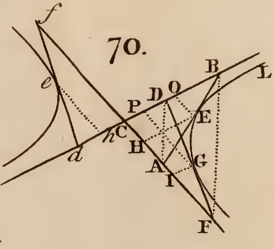

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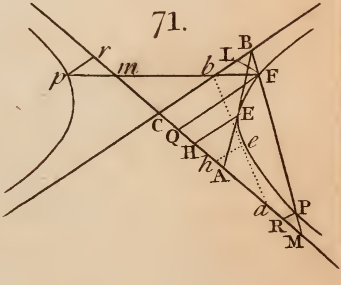

Pl. XX. *pa 138.*

$\overline{HV + VM} = 2HV^2 + 2HV \times VM = CV^2 + 2CV \times VF + VF^2 - VF^2 = \overline{CV + VF}^2 - VF^2 = CF^2 - VF^2 = CF^2 - MF^2$. Fig. 72.

Cor. 1. *In a right angled hyperbola; the aſſymptotes make a right angle with each other.*

For ſince CA = AD, the angle ACD = CDA = $\frac{1}{2}$ a right angle. And twice ACD = a right angle = the angle of the aſſymptotes.

Cor. 2. *If* SB *be drawn perp. to the aſſymptote from the focus*; CB = CA.

For the triangles CAD, CBS, are iſoceles, and ſimilar; and CD = CS; therefore CB = CA.

PROP. LXXIX.

73. *If thro' the given point* G *in the curve, the line* GA *be drawn parallel to the aſſymptote* CH. *And any line* GL *be drawn cutting the curve and aſſymptote in* F *and* L; *and if* FK *be drawn parallel to the aſſymptote* CB. *Then* KL *will always be equal to the given line* AG.

Produce FG to D; then the triangles DGA, FLK are ſimilar and equal, for the reſpective ſides are parallel; and (Prop. LXXII.) GD = FL; therefore AG = KL.

Cor. *If from* F *in the curve,* FK *be drawn parallel to the aſſymptote* CB; *then* FK = AD, *the part of the other aſſymptote.*

PROP.

Fig.

PROP. LXXX.

74. *If* CA, CD *be the ſemiaxes; the aſſymptote* CF, *biſſects the line* DA, *drawn between the ends of the two axes; and all lines* HK *parallel to it, contained between the conjugate hyperbolas* DH, AK.

For draw AE perp. to AC, and equal to CD; and draw DE; then DCAE is a rectangled parallelogram. But (Def. XI.) the aſſymptote CF paſſes thro' E, and (Geom. III. 2.) the diagonals CE, AD biſſect one another in O; therefore DO = AO.

Again, ſince DA, HK are parallel to the aſſymptote CG; therefore (Prop. LXXV.) CI × IK = CO × OA = CO × OD = (LXXV.) CI × IH; therefore IK = IH.

Cor. 1. *If* DA *joining the ends of the two axes, cut the aſſymptote in* O; *then* $4CO^2 = DA^2 = CA^2 + CD^2$.

For ACD is a right angle, therefore (Geom. II. 20. Cor. 6.) OC = OD = OA, and $CO^2 = DO^2 = AO^2$, and $4CO^2 = 4DO^2 = 4AO^2 = DA^2 = CA^2 + CD^2$.

Cor. 2. *The rectangle of the ſides of the inſcribed parallelogram, is equal to the fourth part of the ſum of the ſquares of the two ſemiaxes.*

For draw AP parallel to CE, then COAP is an inſcribed parallelogram, and CO × CP or CO^2 the rectangle of the ſides; and that is $= \frac{CA^2 + CD^2}{4}$.

PROP.

PROP. LXXXI.

If AB *be any diameter*; CG, CL, *aſſymptotes*; GH *a tangent at* B. *Then* GH *is equal to the conjugate diameter of* AB. 75.

For let KF be a tangent at A, CD the ſemiconjugate. Then, becauſe (LXXV. Cor. 3.) the aſſymptote FG is a tangent at an infinite diſtance, therefore (Prop. XXXIV.) $AK \times BH = CD^2$; but the triangles CBH, CAK are equal and ſimilar, and $AK = BH$; therefore $BH^2 = CD^2$, and $BH = CD$, and GH or $KF = 2CD$ the conjugate of AB.

Cor. 1. *If* CG *be an aſſymptote*, BG *a tangent at* B. *Then* CB, BG *are ſemiconjugates to each other.*

Cor. 2. *Hence the further any diameter* AB *is from the axis*; *the greater it is, and the greater its conjugate* GH.

PROP. LXXXII.

If F *be the focus, and it be* CF : CA : : CA : CT, *and* TE *be perp. to* CA; *and if* MF *be drawn from any point of the curve to the focus, and* MN *parallel to the aſſymptote* CO. *Then* $MN = MF$. 76.

Draw ME parallel to CA, and AL perp. to it, then (Def. 6.) $CL = CF$, and the triangles MEN, CAL are ſimilar, and MN : ME : : CL or CF : CA : : (XXVI.) FM : ME. Therefore $FM = MN$.

Fig.

PROP. LXXXIII.

77. *If* F *be the focus, and* FP *perp. to* CA, *and from* M *any point in the curve,* MP *be drawn parallel to either aſſymptote, to cut* FI *in* P. *Then* FM − MP $= \frac{1}{2}$ *latus rectum*; *or* FM + MP, *when* M *is on the other ſide of* I.

For if CF, CA, CT be in geom. progreſſion, and TN perp. to CA. Then (LXXXII.) FM = MN, and FI = IQ = (by the parallels) PN; therefore NM — MP = PN = FI; that is, FM — MP = FI.

PROP. LXXXIV.

78. *If* CF, CB *be aſſymptotes, and* DA *be drawn parallel to the aſſymptote* CB, *cutting the conjugate hyperbolas in* D *and* A; *then* CD, CA *drawn from the center to the interſections, will be conjugate diameters.*

Thro' A, D draw the tangents FG, FL. Then (LXXX.) ſince AO = DO, and (LXXII. Cor. 2.) FA $= \frac{1}{2}$ FG; therefore AO or OD $= \frac{1}{2}$ CG or $\frac{1}{2}$ LC, and DA = CG or CL. Whence (Geom I. 5. Cor. 3.) CD is parallel to AG or FG. And likewiſe CA is parallel to LD or LF; and therefore (Def. 7.) CA, CD are conjugate diameters.

Cor. 1. *Hence it appears that the two conjugate diameters* NA, PD, *belong to the conjugate hyperbolas* AM, DH; *and alſo to the oppoſite hyperbolas* KN, PQ. *And that all theſe four conjugate hyperbolas, make but, as it were, one figure* BAMHDIKNPQ.

Cor. 2. *Hence alſo the conjugate hyperbola* DH *paſſes thro' the ends* D, *of all the conjugate diameters*

PD; *as well as the hyperbola* AM *paſſes thro' all the* Fig.
ends A, *of the firſt diameters* NA. 78.

Cor. 3. *In an equilateral hyperbola, every diameter is equal to its conjugate.*

For then (fig. 73.) CA = CD, and therefore CO is perp. to DA, and to all the parallels HK. Or (in fig. 77.) CO is perp. to DA, and therefore CD = CA.

PROP. LXXXV.

If LKE *be drawn parallel to the aſſymptote* CG, 79.
and any line DV *be drawn, cutting this line, the curve, and aſſymptotes in* E, F, T, D, V; *and the tangent* GBQ *be drawn parallel to* DV, *and* CB *the diameter.*

Then BG : CG : : *rectangle* FET : 2KE × EV.

Draw LP parallel to GQ. Then (LXXVI. Cor. 1.) 4CL × LK = GC × CQ. And by similar triangles CL : LP : : CQ : QG : : CQ × CG or 4CL × LK : QG × CG. Whence CL × CG × QG = LP × 4CL × LK, and CG × QG = 4LP × LK = CG × 2BG.

Then (LXXIV.) LKM : DFV : : KEN : FET. But KM = EN = infinitely; therefore LK : KE : : DFV : FET. And LK × FET = KE × DFV = KE × BG^2. And KE × BG^2 × CG = LK × FET × CG, or KE × BG × 2LP × LK = LK × FET × CG. And 2KE × BG × LP = CG × FET, and BG : CG : : FET : 2KE × LP or 2KE × EV.

Cor. 1. CG × BG = 2LP × LK.
From the demonſtration.

Cor. 2. *If more parallels be drawn it will be* FET : *fet* : : KEV : *kev.*

Cor.

Fig. 79. Cor. 3. *If two parallels meet one line* LE; *then* FET : *fot* : : KE : KO.

Cor. 4. *Hence alſo fet* : *fot* : : *kev* : K*ov*.

PROP. LXXXVI.

80. *If* FBE *be an hyperbola between the aſſymptotes* C*f*, CL; *and it be* CG : CH : : CK : CL; *and lines or ordinates be drawn parallel to the other aſſymptote* C*f*. *Then the area* GHDF = *area* KLEB.

The inſcribed rectangles being equal (Prop. LXXV.), their ſides are reciprocally proportional, EL : BK : : (CK : CL : : (hypoth.) CG : CH : :) DH : FG; therefore if any lines CG, CH, CK, CL are proportional; the ordinates GF, HD, KB, LE will alſo be proportional, and in the ſame ratio inverſly.

Divide GL into an infinite number of equal parts at I, P, G, R, &c. in geometrial progreſſion; and draw the ordinates IM, PN, QO, RS, &c. parallel to GF or C*f*; let the ſame be continued to L, and let r to s, be the ratio of the terms of the progreſſion. Whence we ſhall have

r : s : : CG : CI : : CI : CP : : CP : CQ : : CQ : CR, &c. and (Propor. XXII.) r : s : : GI : IP : : IP : PQ : : PQ : QR, &c. and by what went before

s : r : : IM : PN : : PN : QO : : QO : RS, &c. and multiplying the two laſt,

rs : rs : : GI × IM : IP × PN : : IP × PN : PQ × QO : : PQ × QO : QR × RS, &c. therefore the infinitely ſmall parallelograms are equal; that is, GIM = IPN = PQO = QRS, &c. = KVA = VTW = TXZ = XY*a*, &c.

Now, in the progreſſion CG, CI, CP, CQ, &c. if we take any equidiſtant terms CG, CH, and CK, CL (Propor. XXI.) theſe ſhall be proportional; that is, CG : CH : : CK : CL. And the terms

CH, CL, being equidiſtant from CG, CK; there is the ſame number of parallelograms between G and H, as between K and L. And theſe parallelograms being all equal, the ſum of all between G and H, is equal to the ſum of all between K and L. But by the method of indiviſibles, the ſum of all the parallelograms GIM, IPN, &c. compoſe the area GHDF; and the ſum of all the parallelograms KA, VW, &c. compoſe the area KLEB; therefore the area GHDF = area KLEB. Fig. 80.

81. Cor. 1. *If* CF, CD *be drawn, and alſo* DH, FG *parallel to the aſſymptote* C*f*. *Then the ſector* CFD = *area* FGHD.

For ſince rectangle CGF = CHD, triangle CGF = CHD; take away CG*q* common, then C*q*F = G*q*DH. Add F*q*D common, then C*q*F + F*q*D or ſector CFD = G*q*DH + F*q*D = FGHD.

Cor. 2. *If it be* CG : CH : : CK : CL; *then ſector* CFD = *ſector* CBE.

For CFD = FGHD, and CBE = BKLE.

Cor. 3. *If* CK : CL : : C*d* : C*f*. *And* BK, EL *be drawn parallel to* C*f*; *and* D*d*, F*f* *to* CL; *then ſector* CBE = CDF = *area* BKLE = D*df*F.

For CK : CL : : C*d* : C*f* : : CG : CH. Whence ſector CFD = D*df*F = FGHD, &c.

Cor. 4. *Hence the ſpace* D*df*F = *ſpace* DHGF.

For the parallelogram *d*G added to each makes the ſame ſpace C*f*FDH.

Cor. 5. *If* CG, CH, CK, CL *are in geometrical progreſſion*; *then ſector* CFD, CDB, CBE *are equal*; *and alſo the ſpaces* FGHD, DHKB, BKLE.

Cor. 6. *If* CL *be the nth term in geometrical progreſſion, in the ratio of* CG *to* CH; *then ſector* CFE $= \overline{n-1} \times$ CFD; *and* FGLE $= \overline{n-1} \times$ FGHD.

For

Fig. 81. For if CH is the ſecond term, then CFD = 1 × CFD. If CK be the third term, then CFB = CFD + CDB = 2 × CFD. If CL be the fourth term, CFE = FCD + DCB + BCE = 3 × CFD, and ſo on. And the like for the areas FGHD, FGKB, &c.

Cor. 7. *If* CG, CH, CK, CL, *are proportionals; then* LE, KB, HD, GF *are alſo proportionals, and in the ſame ratio, or* FG, DH, BK, EL *are proportionals, in the inverſe ratio.*

From the demonſtration.

SCHOLIUM.

Hence if CG = 1, *and* CH, CK, CL, *&c. be any numbers; the (ſectors or) hyperbolic ſpaces* GFDH, GFBK, GFEL, *&c. are analogous to the logarithms of theſe numbers.*

For (Cor. 6.) whilſt the numbers CG, CH, CK, CL, &c. proceed in geometrical progreſſion, the correſpondent (ſectors or) ſpaces, proceed in arithmetical progreſſion; and therefore (from the nature of logarithms) are proportional to the logarithms of theſe numbers.

82. Make CG = CN = 1. Then by computation from infinite ſeries, &c. if the number CH be 10, the ſpace GFDH or log. 10. will be found 2.30258509. If the number CK be 100, the ſpace GFBK or log. 100 will be 4.60517018. If the number CL be 1000; the ſpace GFEL or log. 1000, will be 6.90775527. And if the hyperbolic ſpaces correſponding to the intermediate numbers be likewiſe computed; they will be the logs. of theſe numbers. And theſe logarithms, becauſe they are computed from the hyperbolic areas, are called *hyperbolic Logarithms.*

PROP.

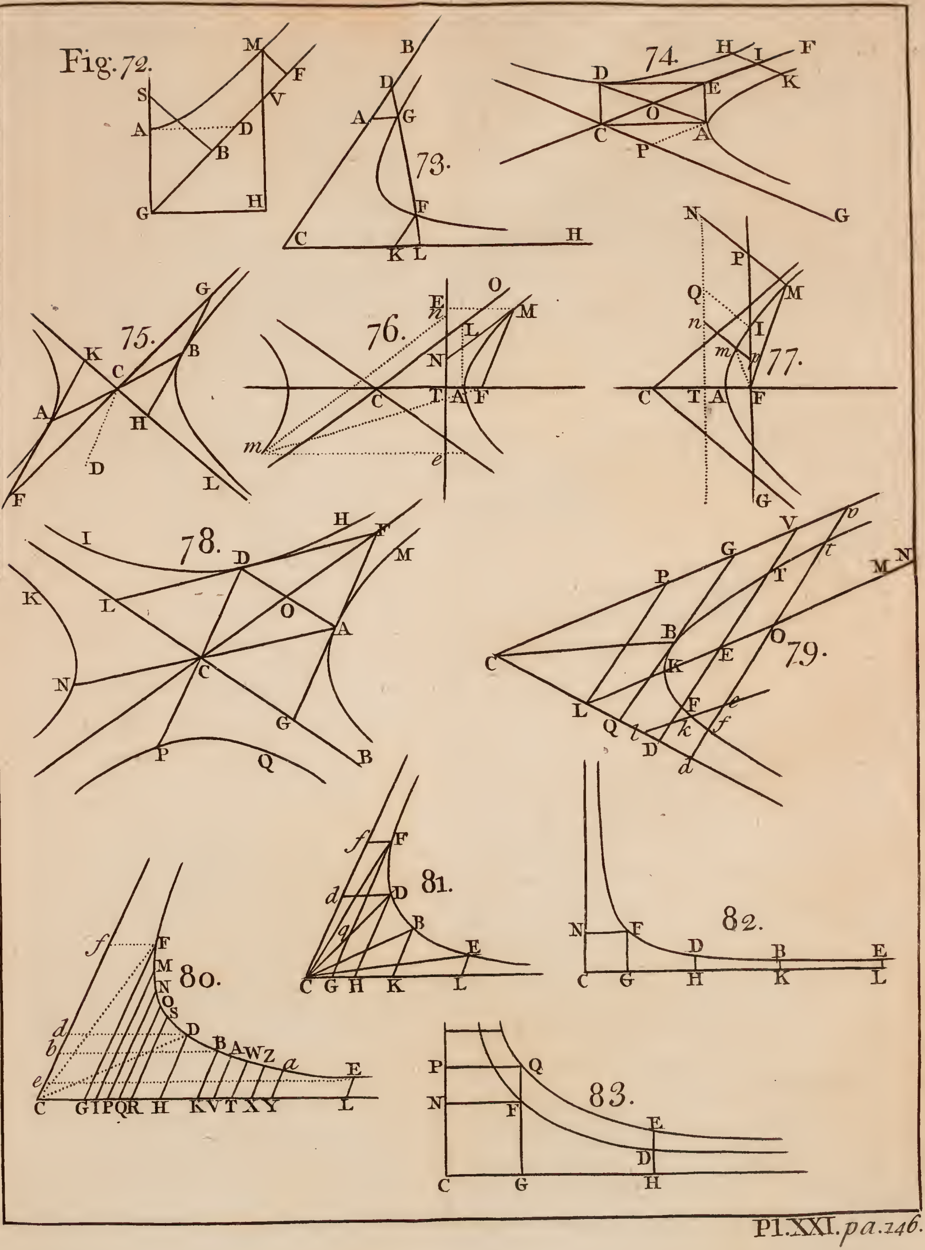
Fig. 72.
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Pl.XXI. *pa.146.*

PROP. LXXXVII.

If two hyperbolas QE, FD, *have the ſame aſſymptotes* CP, CH; *if* HDE *be drawn parallel to the aſſymptote* CP; *then* HD *is to* HE *every where in the given ratio of* CN *to* CP. 83.

For the inſcribed rectangles are CG × GF, and CG × GQ. Therefore (LXXV.) CH × HD = CG × GF, and CH × HE = CG × GQ. Whence GF : GQ : : CGFN : CGQP : : CH × HD : CH × HE : : HD : HE.

Cor. 1. *The area* FGHD : *area* QGHE : : *ordinate* GF : *ordinate* GQ : : *or as inſcribed parallelogram* CF : *inſcribed parallelogram* CQ.

For the ordinate HD to HE is every where as CN to CP. Therefore if the baſe GH be divided into an infinite number of equal parts, and ordinates HDE erected. It will be GF : GQ : : HD : HE : . ſum of all the HD . ſum of all the HE : : area FGHD : area QGHE.

Cor. 2. *If* CG = GS = 1, *and* GF = .43429448; *then the area* FGHD *is the common logarithm, or Brig's logarithm of the number* CH. 84.

For the hyp. logarithm is to the common logarithm, as 2.30258509 to 1, or as 1 to .43429448, or as GS to GF; that is, (Cor. 1.) as the area GSEH to GFDH.

SCHOLIUM.

Hence Brig's logarithms will be had by computing the hyperbolic areas GFDH, whoſe inſcribed parallelogram is .43429448. Thus if CG = 1, CH = 10, the area FGHD = 1; if CH = 100, the area FGHD = 2. If CH = 1000, FGHD = 3, and ſo on; and the like for the intermedi-

Fig. 84. ate numbers. Likewiſe ſince the area at G is nothing; therefore the log. of CG or 1 is 0; and for numbers leſs than CG the area lies on the other ſide of SG; and therefore the logarithms of numbers leſs than 1, are negative. And ſince the area SGCB is infinite, therefore the logarithm of 0, is an infinite negative.

It is plain likewiſe that logarithms may be of as many different forms as you pleaſe. For if CG be 1, then the log. GSEH will be as the inſcribed parallelogram CRSG. Therefore as that parallelogram is greater or leſs, the logarithms of numbers will be proportionally greater or leſs. And therefore in different ſpecies of logarithms, the logarithms of particular numbers will be in a given ratio.

On the contrary, if you have the logarithms of numbers, the hyperbolic area may be computed. For it will be as .43429, &c. to the inſcribed parallelogram CGSR : : ſo log. of CH, to the area GHES or ſector CSE.

PROP. LXXXVIII.

85. *If* AN *be an equilateral hyperbola,* AM *any hyperbola with the ſame tranſverſe axis. The area* ANP *of the equilateral hyperbola : area* AMP *of the other hyperbola : : as the ſemitranſverſe* CA *: to ſemiconjugate* CD.

For if CE be the ſemiconjugate of AN, then $CE = CA$. And (Prop. LXVII.) $CP^2 - CA^2 = PN^2$. And (Prop. VI) $CA^2 : CD^2 : : CP^2 - CA^2$ or $PN^2 : : PM^2$. And CA : CD : : PN : PM. Therefore if the abſciſſa AP be divided into an infinite number of equal parts and ordinates erected; it will always be CA : CD : : PN : PM : : ſum of all the PN : ſum of all the PM : : area ANP : area AMP.

Cor.

Fig. 85. Cor. 1. *The areas of two hyperbolas on the ſame tranſverſe, and having the ſame abſciſſa*, AP; *are as their conjugates.*

86. Cor. 2. *As ſemitranſverſe* CA : *ſemitranſverſe* CB : CANP *the area of the hyperbola* AN : CBMP *the area of the hyperbola* BM, *on the ſame abſciſſa* CP.

By reaſoning as before $CD^2 : CD^2 + CP^2 :: CA^2 : PN^2 :: CB^2 : PM^2$; and CA : CB : : PN : PM : : all the PN : all the PM : : area CANP : area CBMP.

Cor. 3. *Therefore if the area of an equilateral hyperbola be known; the area of any other hyperbola may be found.*

PROP. LXXXIX.

87. *If* NAM *be a hyperbolic conoid, made by revolving round the tranſverſe axis. If* C *be the center,* BA *the tranſverſe; then* $2CA + AP : CA + \frac{1}{3}AP ::$ *circumſcribing cylinder : ſolidity of the conoid* NAM.

Let ſemitranſverſe CA $= t$, ſemiconjugate CD $= c$, AP $= x$, PM $= y$, $p = 3.1416$. Then (Prop. VI.) $tt : cc :: 2tx + xx :: yy$, and $yy = \frac{tt}{cc} \times \overline{2tx + xx}$. And $pyy = \frac{ptt}{cc} \times \overline{2tx + xx}$, for the circle NM. Therefore if AP be divided into an infinite number of equal parts, according to the method of indiviſibles; and planes drawn thro' all theſe points, all theſe circular planes will make up the ſolid. Now (Arith. Infinites Prop. II.) the ſum of all the $2tx$ is $= \frac{2txx}{2} = txx$, and (ib. Prop. III.) the ſum of all the xx is $= \frac{x^3}{3}$; there-

Fig. 87. fore the ſum of all the $pyy = \frac{ptt}{cc} \times \overline{txx + \frac{x^3}{3}} = \frac{pttx}{cc} \times \overline{tx + \frac{1}{3}xx}$. And the circumſcribing cylinder $= pyyx = \frac{pttx}{cc} \times \overline{2tx + xx}$; therefore the cylinder : to the ſolid : : $2txx + xx : tx + \frac{1}{3}xx :: 2t + x : t + \frac{1}{3}x$.

Cor. 1. *The ſolidity of the hyperboloid* : *to a cone of the ſame baſe and altitude* : : BP : BP + AC.

For the cone is $\frac{1}{3}$ of the cylinder; and it is, cylinder : ſolid : : $2t + x : t + \frac{1}{3}x$; therefore cone : ſolid : : $2t + x : 3t + x$.

Cor. 2. *If* AP = AC, *the conoid* NAM = $\frac{4}{9}$ *the circumſcribing cylinder.*

If AP = AB, *the conoid* = $\frac{5}{12}$ *the cylinder.*

If AP = 3AC, *the conoid* = $\frac{2}{5}$ *the cylinder.*

If AP = 2AB, *the conoid* = $\frac{4}{9}$ *the cylinder, &c.*

Cor. 3. *The hyperbolic conoid* NAM, *is equal to the fruſtum of the aſſymptotic cone* EIAKF, *leſſened by the cylinder* GIAKH.

For (Prop. LXIX) the rectangle EMF = AK^2; therefore (Geom. IV. 38.) the annulus or ring contained between the two circles NM and EF is equal to the circle IK. Therefore if AP be divided into an infinite number of equal parts; all the rings compoſing the hollow ſolid NIAKM = all the circles GH, compoſing the cylinder GIKH. Therefore, fruſtum EIKF — cylinder GIKH = fruſtum EIKF — ſolid NIAKM; that is, = hyp. conoid NAM.

Cor. 4. *The hollow ſolid* NIAKM *incloſing the hyp. conoid is* = *cylinder* GIKH. Fig. 87.

PROP. XC. *Prob.*

To deſcribe a hyperbola by continued motion. 88.

1 *Way.*

Draw the line DB, and let G be a fixed point; fix the two rulers together IK, KL, ſo as to make the given angle IKL; let there be another ruler SL, which moves about the fixed point L in the leg KL, as a center. Then cauſe the ſide KL to move along the line DB, and coincide with it; whilſt the ruler SL ſlides by the fixed point (or pin) G. Then the lines SL and KI will croſs one another; and their interſection F, will deſcribe the hyperbola FOG.

For take SG = FL, and draw SC parallel to IK meeting DB in C. Then G is a fixed point in the curve, and FK is parallel to SC; and GL paſſes thro' L, ſo that KL is always given. Therefore this is a property of the hyperbola, (by Prop. LXXIX.) whence the curve is a hyperbola whoſe aſſymptotes are SC, LC; and center C.

2 *Way.*

Let FL, GH be aſſymptotes, AB a diameter. 89. Let AG be parallel to the aſſymptote FL; take HK = CG, make the angle CHM = GCF. Then move the right line HK along the aſſymptote HG, whilſt the right line AM (moveable about A) paſſes thro' the point K; then the interſection (of HM, AM), M will deſcribe the hyperbolas MB, A*m*.

For the triangles KHM, KGA are ſimilar; whence KH : HM : : KG : GA; that is, CG : HM : : CH : GA; whence CH × HM = CG × GA; therefore (LXXV.) M is in the hyperbola.

Fig. 89.

SCHOLIUM.

The hyperbola may alſo be deſcribed as directed in Def. 1, and 2.

PROP. XCI. *Prob.*

90. *To deſcribe an hyperbola, by finding many points in the curve.*

1 *Way.*

Let BA be the axis; S, F the foci. Take any point N beyond F, and with extent AN, and one foot in F, deſcribe an arch M. Then with extent NB, and one foot in S, croſs the former arch at M. Thus find as many points as neceſſary, and draw the hyperbola thro' them.

By this conſtruction, SM — FM = AB; therefore (Prop. I.) the point M is in the curve.

2 *Way.*

91. Let AB, DG be two diameters conjugate to one another. Divide CA into any number of equal parts at L, L, &c. draw DA, and parallel to it, draw LI, LI, &c. draw CF = CB, and perp. to it. Then thro' I, I, &c. draw MN, MN, &c. parallel to AB; and ſo that IM or IN be equal to LF. Then M, M, &c. and N, N, &c. will be oppoſite hyperbolas.

For by ſimilar triangles $CD^2 : CA^2 :: CI^2 : CL^2 ::$ (compounding) $CD^2 + CI^2 : CA^2 + CL^2$ or IN^2 or IM^2. For $CA^2 + CL^2 = CF^2 + CL^2 = LF^2$. Therefore M and N are in the curves, by Prop. XLIII.

3 *Way.*

92. Let CA or CB be the ſemitranſverſe, CD the ſemiconjugate; produce CD at pleaſure, and draw

DE parallel to CB; draw DA, and from any Fig.
point P in the axis, draw PF parallel to AD, to 92.
cut CD in F. With radius CF, and center C deſcribe a circle to cut DE in G. At P, draw the ordinate PM, perp. to CP, and equal to DG. Then M will be in the curve.

For by ſimilar triangles $CA^2 : CD^2 :: CP^2 : CF^2 ::$ (by diviſion) $CP^2 - CA^2 : CF^2 - CD^2$. But $CP^2 - CA^2 = BPA$; and $CF^2 - CD^2 = CG^2 - CD^2 = DG^2 = PM^2$; therefore $CA^2 : CD^2 :: BPA : PM^2$. Whence (Prop. VI.) the point M is in the curve.

PROP. XCII. *Prob.*

Given the aſſymptotes FG, MN, *and the point* B 93.
in the curve, to deſcribe the hyperbola.

1 *Way.*

Thro' B draw ſeveral right lines to cut the aſſymptotes, as ABN, DBE, GBL, &c. Then make NT = AB, ES = DB, LI = GB, &c. Then the points T, S, I, are in the curve.

Likewiſe thro' any point ſo found as T, you may draw other lines as HTQ, KTO, &c. and make PQ = HT, KT = OR, &c. then R, Q, &c. are alſo in the curve.

This proceſs is plain from Prop. LXXII.

2 *Way.*

Thro' the given point B, draw BA, BD paral- 94.
lel to the aſſymptotes. Then take any length CG, and draw GH parallel to CD, and ſo that $CG : CA :: CD : GH$; then H is in the curve, and ſo for more points.

Or take the equal parts CA, AE, EG, GI, &c. And make EF = $\frac{1}{2}$ AB, GH = $\frac{1}{3}$ AB, IK = $\frac{1}{4}$ AB, and ſo on. Do the ſame in the aſſymp-

Fig. tote CD; then thro' all the points *f*, B, F, H, K,
94. draw the hyperbola required.

PROP. XCIII. *Prob.*

95. *A right line* DE *in the hyperbola being given; to find its diameter, and the center.*

Draw HI parallel to DE; biſſect DE, HI, in F and G; thro' which draw GFAB; then AB is the diameter. Biſſect AB in C, then C is the center.

For the diameter biſſects all the whole ordinates, and is itſelf biſſected in the center.

PROP. XCIV. *Prob.*

95. *Any diameter* AB *in a hyperbola, being given; to find its conjugate.*

Draw any line KI parallel to AB, and biſſect it in L; thro' L and the center C, draw LC, for the conjugate, which will be terminated by the conjugate hyperbolas, if they are there; otherwiſe its magnitude will be determined by Prop. XXXVI.

PROP. XCV. *Prob.*

96. *Any diameter* AB *being given; to draw an ordinate to it, from a given point* H *in the curve.*

Thro' the center C, draw the diameter HK; and from K, draw KL parallel to AB. From L draw LH for the double ordinate.

For ſince HC = CK, and CO is parallel to KL, therefore HO = OL.

Or thus,

Draw HI parallel to AB, biſſect IH in G; thro' G and the center C, draw GC, and HO parallel to it for the ordinate.

For

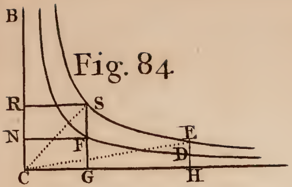
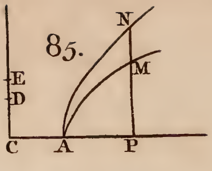


Fig. 84



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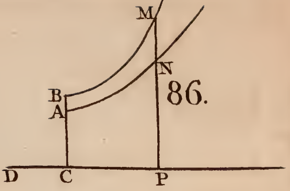

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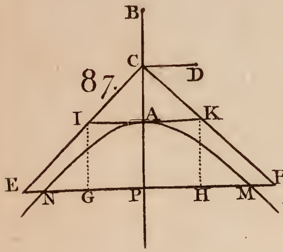

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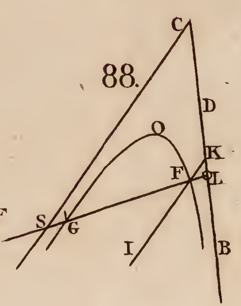

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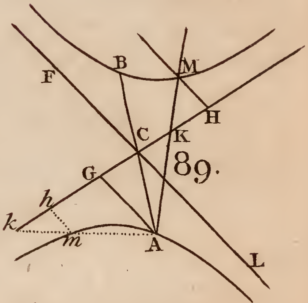

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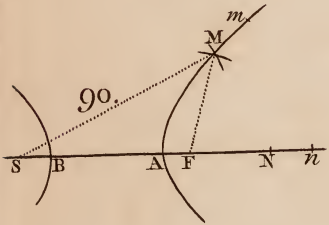

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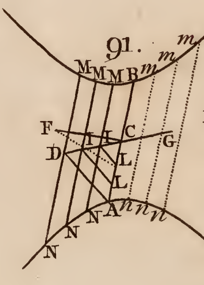

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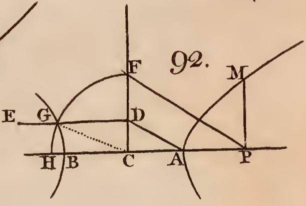

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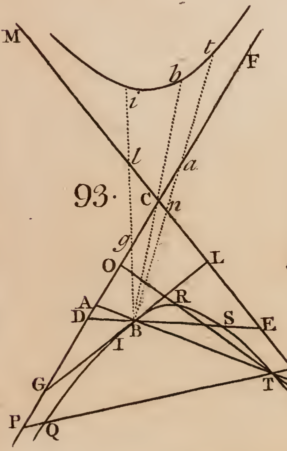

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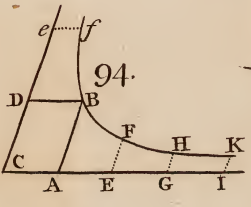

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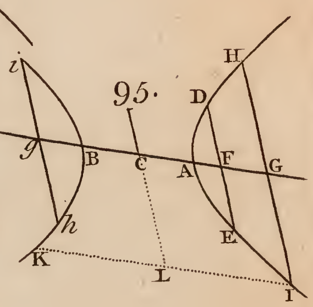


95.

Pl. XXII. *pa. 154.*

For GC, biſſecting IH, is the poſition of the conjugate, and HO being parallel to it is an ordinate to CO. Fig. 96.

PROP. XCVI. *Prob.*

To find the two axes of a given hyperbola. 97.

Draw two parallels thro' the figure, FK, ML; thro' the middle of them draw the diameter AB; biſſect it in C for the center.

On the center C deſcribe an arch, of a circle, to cut the curve in two points G, H. Draw GH and biſſect it in O. Thro' O and the center C, draw CO for the axis. And CD perp. to it for the conjugate.

For AB paſſing thro' C is a diameter, and ſince CG = CH; therefore CO paſſing thro' the middle of GH, will be perp. to it, and being perp. to the ordinate GO, it is the axis.

PROP. XCVII. *Prob.*

A hyperbola being given; to find two conjugate diameters, making a given angle with one another. 98.

About any diameter AB, deſcribe the arch of a circle ADB, to contain an angle equal to the given one, or its ſupplement. To the point F where it cuts the hyperbola, draw BF, AF; which biſſect in O and L; and from the center C, draw CO, CHL. Then CH and CO (terminating at the conjugal hyperbola,) will be the two axes.

For ſince AL = $\frac{1}{2}$ AF, and AC = $\frac{1}{2}$ AB, CL is parallel to BF. And ſince FO = $\frac{1}{2}$ FB, CO is parallel to AF; whence LCO = OFL = the given angle.

SCHOLIUM.

If CE be the ſemiconjugate of AC, and BE be drawn and HG parallel to it, cutting CO in G; then

Fig. then CG is the conjugate, and G a point in the
98. conjugate hyperbola. For by Prop. LXXXIV. BE, GH being both parallel to the aſſymptote; CH, CG are conjugates, as well as CB, CE.

PROP. XCVIII. *Prob.*

99. *Given the aſſymptotes and a point in the curve; to find two conjugate diameters.*

From the given point B, draw BHD parallel to the aſſymptote GC, make HD = HB. Draw DCE, and make CE = CD, then DE is the conjugate of AB.

For draw BF parallel to CD, then the triangles CDH, FBH are ſimilar; and they are equal, becauſe BH = HD, and therefore CH = HF; and conſequently GB = BF. And therefore FG is a tangent at B (LXXII. Cor. 2.). And ſo FG is equal to the conjugate of CB (LXXXI.); whence DE equal and parallel to FG, is its conjugate.

Cor. *Hence if two conjugate diameters be given, and the angle they make with one another, be given; the aſſymptotes may be drawn. Which is done by drawing* CG *parallel to* DB, *and* CF *parallel to* EB.

PROP. XCIX. *Prob.*

To draw a tangent to any point M *in the hyperbola.*

1 *Way.*

100. Draw the ſemidiameter MC, and find its conjugate CD (XCIV.); then from M, draw MT parallel to CD, which will touch the curve in M.

2 *Way.*

101. Let BA be any diameter, draw MP an ordinate to it; and make CP, CA, CT, continual proportionals,

tionals; then draw MT which will be a tangent Fig.
at M. By Prop. XLII. 101.

3 *Way.*

Draw the lines MF, MS, from the point M to the focus's; make ME = FF, draw FE, which biſſect in D, then MD drawn will be a tangent at M.

Or produce SM to G, ſo that MG = MF. Draw FG, and MT parallel to it, for the tangent at M.

For in the iſoceles triangle EMF, ED = DF; therefore $<$ EMD = FMD, whence (Prop. IX.) MD is a tangent And ſince MD is parallel to GF, and MF = MG; therefore EMD = EGF, and EMF = 2EGF, and $\frac{1}{2}$ EMF = EGF = EMD, therefore MD is a tangent.

4 *Way.*

Let CH, CK be the aſſymptotes; from the point 102.
M draw MD parallel to one aſſymptote CH. Make DF = DC; and draw FM which will be a tangent at M.

For produce FM to G; then ſince FD = DC; therefore (by the parallels CG, DM) FM = MG; whence (LXXII. Cor. 2.) MF is a tangent.

PROP. C. *Prob.*

From a given point without the hyperbola, to draw a tangent to it.

1 *Way.*

Let T be the point given. Thro' T draw the 101.
diameter BA. Make CT, CA, CP, continual proportionals. At P draw the ordinate PM, to the diameter AB. From M to T draw MT for the tangent. By Prop. XLII.

2 *Way.*

Let T be the point given. Thro' T draw the 103.
diameter AB; and find its conjugate CD (XCIV); draw

Fig. draw any line TS = TB, and make TR = TC;
103. draw AR, and SP parallel to it, cutting the diameter AB in P. Draw the ordinate PM parallel to CD; then TM is the tangent.

For by ſimilar triangles TS : TR : : TP : TA; that is, by conſtruction, TB : TC : : TP : TA. Therefore (XLII. Cor. 3.) TM is a tangent.

PROP. CI. *Prob.*

104. *The focus* F *being given, and three points in the curve; to deſcribe the hyperbola.*

Thro' the two points N, M, draw NMK, and alſo thro' N, P, draw NIP. Draw MF, NF, PF to the focus. And make FN : FM : : NK : MK, and FN : FP : : NI : PI. Thro' K and I, draw KG; draw FT, MH perp. to it. Take the points A, B, ſo that FM : MH : : FA : AT : : BF : BT; then BA is the axis, and C the middle point, is the center. Make BS = AF, and S is the other focus; and the hyperbola may be deſcribed by Prop. XCI.

Draw NG, PL perp. to KT. Then by conſtruction, FN : FM : : NK : MK : : (ſimilar triangles) NG : MH, whence FN : NG : : FM : MH. Alſo FN : FP : : NI : PI : : NG : PL, whence FN : NG : : FP : PL. Therefore FN : NG : : FP : PL : : FM : MH : : (conſtruction) FA : AT : : BF : BT. And this (Prop. XXVI.) is the property of the hyperbola, where TG is the directrix. For, ſince AT : AF : : BT : BF : : (compounding) BT + AT . BF + AF : : BA : SF : ; CA : CF; therefore (XXVI. Schol.) TG is the directrix.

PROP. CII. *Prob.*

Given two conjugate diameters of a hyperbola; CH, CL, *to find the two axes.* 105.

Join the ends of the diameters with the line LH, which biſſect in O, draw the aſſymptote COI. Draw CF parallel to LH, and CF is the other aſſymptote. Draw CP to biſſect the angle HCF, and CQ perp. to it. Then the axes lie in the lines CP, CQ.

Make IO = CO, and thro' H, L, draw IK, IG. Alſo draw HP, LQ perp. to CP, CQ. And take CA a mean proportional between CK and CP; alſo CE a mean between CG, CQ. Then CA, CE are the ſemiaxes.

For (LXXX.) ſince LO = OH, CO is one aſſymptote, and CF (parallel to LH) is the other; and ſince CO = OI, therefore (LXXII. Cor. 2.) IK, IG are tangents. And ſince CK : CA : CP are in geometrical progreſſion; therefore (Prop. XV.) CA is the axis, and it biſſects the angle HCF. And ſince CG : CE : : CE : CQ, and CE perp. to CA; CE is the other axis.

PROP. CIII. *Prob.*

To deſcribe a hyperbola about a given parallelogram EGHF, *to paſs thro' a given point* P. 106.

Let the conjugate diameters AB, ID biſſect the oppoſite ſides of the parallelogram; draw PQ parallel to ID; and take $AC^2 = \dfrac{CQ^2 \times EL^2 - CL^2 \times PQ^2}{EL^2 - PQ^2}$. And $CD^2 = \dfrac{CQ^2 \times EL^2 - CL^2 \times PQ^2}{CL^2 - CQ^2}$.

For

Fig. 106. For (Prop. XXXVI.) $CA^2 : CD^2 :: ALB : EL^2 :: AQB : PQ^2$. That is, $AC^2 : CD^2 :: CL^2 - CA^2 : EL^2 :: CQ^2 - CA^2 : PQ^2$. Hence $\overline{CL^2 - CA^2} \times PQ^2 = \overline{CQ^2 - CA^2} \times EL^2$; therefore $CA^2 \times \overline{EL^2 - PQ^2} = CQ^2 \times EL^2 - CL^2 \times PQ^2$, which determines AC.

Again, ſince $AC^2 : CD^2 :: CL^2 - CA^2 : EL^2$; by compounding, $AC^2 : CD^2 :: CL^2 : CD^2 + EL^2$. And ſince $CL^2 - CA^2 : EL^2 :: CQ^2 - CA^2 : PQ^2$; and (by diviſion) $CL^2 - CA^2 : EL^2 :: CL^2 - CQ^2 : EL^2 - PQ^2$. Therefore $CL^2 : CD^2 + EL^2 :: CL^2 - CQ^2 : EL^2 - PQ^2$. And (dividing) $CL^2 : CD^2 + EL^2 :: CQ^2 : CD^2 + PQ^2$. Whence $CL^2 \times \overline{CD^2 + PQ^2} = CQ^2 \times \overline{CD^2 + EL^2}$; and $CD^2 \times \overline{CL^2 - CQ^2} = CQ^2 \times EL^2 - CL^2 \times PQ^2$; which determines CD.

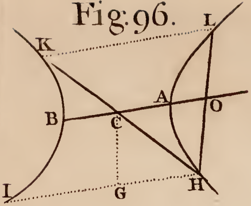
Cor. $AC^2 : CD^2 :: CL^2 - CQ^2 : EL^2 - PQ^2$. For $AC^2 \times \overline{EL^2 - PQ^2} = CQ^2 \times EL^2 - CL^2 \times PQ^2 = CD^2 \times \overline{CL^2 - CQ^2}$.

PROP. CIV.

107. *If a cone* MDN *be cut by a plane, parallel to any plane* DXY *within the cone, which paſſes thro' the vertex* D; *the ſection* LAT *will be an hyperbola.*

Let the plane produced cut the oppoſite cone BD*p* in B*tl*; biſſect AB in C, and parallel to the baſe MN, draw PTQ, CGKF, *ptq*, whoſe ſections, with the cone will be circles; and the interſections with the plane of the hyperbola, will be LT, *lt*. Draw PQ, *pq* perp. to LT, *lt*; and CK a tangent to the circle GKF; then the rectangle PIQ $= LI^2$, and rectangle *piq* $= li^2$, and rectangle GCF $= CK^2$. The plane PDQ interſecting the other planes, will

Fig: 96.



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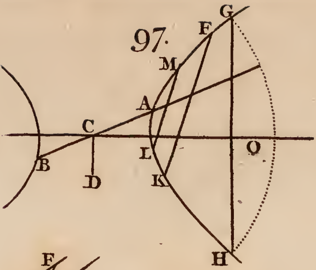

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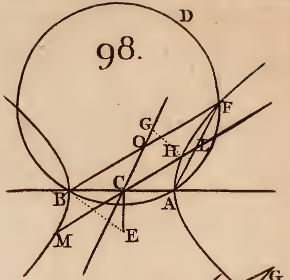

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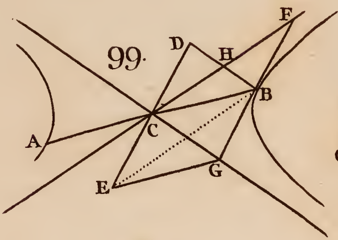

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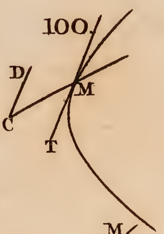

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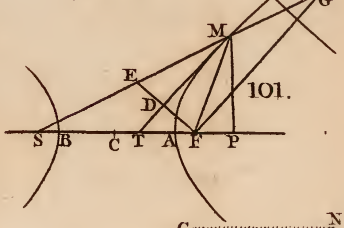

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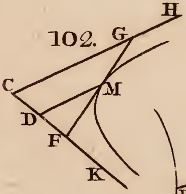

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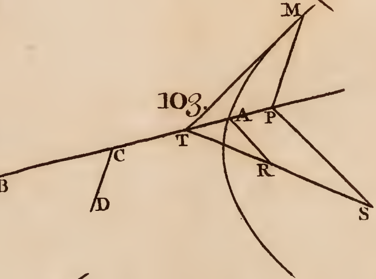

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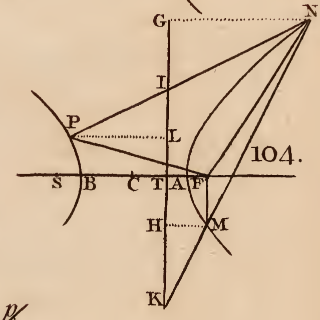

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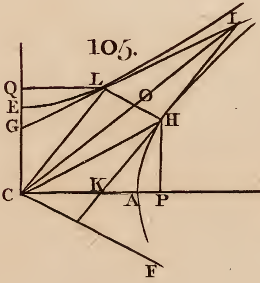

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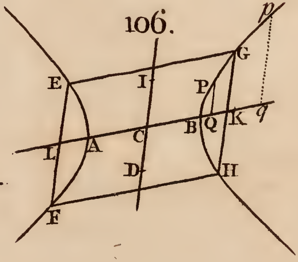


Pl. XXIII. *pa. 160.*

Fig. 107. will make the ſimilar triangles AIP, ACG; and alſo BIQ, BCF. Whence

$AI : IP :: AC : CG$

and $BI : IQ :: BC : CF$. And multiplying,

$AI \times BI : PI \times QI :: AC \times BC : CG \times CF$.

that is, $AIB : LI^2 :: AC^2 : CK^2$.

Therefore (Prop. XXXVI.) the ſection LAT is a hyperbola.

Cor. 1. *The ſame plane cutting the oppoſite cone, the ſection will be the oppoſite hyperbola* tBl, *equal and ſimilar to* TAL.

For $Ai : ip :: AC : CG$, and $Bi : iq :: BC : CF$. Whence $AiB : piq :: AC \times BC : CG \times CF$; that is, $AiB : il^2 :: AC^2 : CK^2$; therefore when $Bi = AI$; then $li = LI$, and the curves are both alike.

Cor. 2. *The ſquare of the conjugate*, $4CK^2 = BH \times AE$, *the rectangle of the diameters of the fruſtum.*

For the circles BH, AE being parallel to the baſe; ſince $AC = \frac{1}{2} AB$, therefore $CG = \frac{1}{2} BH$, and $CF = \frac{1}{2} AE$; therefore CK^2 or $GC \times CF = \frac{1}{2} BH \times \frac{1}{2} AE$.

Cor. 3. *The latus rectum* $= \dfrac{BH \times AE}{AB}$.

For it is a third proportional to the tranſverſe and conjugate.

Cor. 4. *The hyperbola is right angled when* $CK = CA$, *or when* $BH \times AE = AB^2$.

108. Cor. 5. *Thro' the vertex of the cone* D, *draw the plane* DXY *parallel to the plane of the hyperbola* ALT; *and thro'* X *and* Y, *draw the planes* DMX, DMY, *touching the cone in the lines* DX, DY; *and interſecting the plane of the hyperbola in the lines* NCn, OCo; *then theſe lines* NCn, OCo, *are the aſſymptotes of the oppoſite hyperbolas* LAT, lBt; *and* YO *or* XN, *the ſemiconjugate*; *and the angle* NCO = *angle* XDY; *and* MD *paſſes thro' the center* C.

For

Fig. 108. For thro' A draw the circle AGE*s* parallel to the baſe of the cone, which will touch the plane of the hyperbola in A. This will cut the planes DMN, DMO, in the lines *sr*, G*w*. And then the lines *sr*, *r*A, A*w*, *w*G are all tangents to the circle; therefore *rs* = *r*A, and A*w* = *w*G. Likewiſe *r*A = A*w*, becauſe the parallels thereto NI = IO. And ſince *sr*, G*w*, are parallel to XN, YO; therefore XN = *sr* = *r*A = A*w* = *w*G = YO; and therefore XN^2 or rA^2 = rectangle LNT, by the property of the circle; and conſequently (LXXI. Cor. 1.) CN, CO, are aſſymptotes; and (LXXXI.) *rw* is the conjugate, and *r*A = XN or YO is the ſemiconjugate.

Draw PK parallel to AB; then by Prop. XVI. Cor. 2. ellipſis, which alſo agrees with the circle, QZ : PZ : : QM : PM : : (by diviſion) QM — QZ or MZ : PM — PZ. And alternately, QZ : MZ : : PZ : PM — PZ : : (by addition) QZ + PZ or PQ : (MZ + PM — PZ) 2PM. And alternately, QZ : PQ : : MZ : 2PM. But by ſimilar triangles,

QZ : PQ : : ZD : PK.
and MZ : PM : : ZD : PF.
or MZ : 2PM : : ZD : 2PF.
whence ZD : PK : : ZD : 2PF.

Therefore PK = 2PF, whence KP is biſſected in F, and conſequently AB (parallel to it) is biſſected in C, and ſo C is the center.

Laſtly, ſince CN, CO, are parallel to DX, DY; the angle NCO = angle XDY.

109. Cor. 6. *If* AI *be perp. to the baſe of the cone, and* AB *perp. to* MH *the tranſverſe of the hyperbola. And if* AN = AB, *and* FND *be drawn parallel to the baſe* PQ. *Then* FD *is equal to the latus rectum. And* DS *(parallel to* MH*)* = HZ *the diameter at the vertex* H.

Draw

Fig. 109. Draw AL, AG parallel to PQ, MH, and put R = latus rectum. Then the triangles ABL, ANG, are ſimilar and equal, whence AG = AL. The triangles MCH, DFS, GFA, are ſimilar; whence MC : MH : : DF : DS : : GF : GA or AL : : (by the ſimilar triangles GFA, LAH) AF : AH : : (triangles AFD, AHZ ſimilar) FD : HZ. Whence DS = HZ. And MH × FD = MC × HZ = (Cor. 2.) ſquare of the conjugate = (IV. Schol.) MH × R; therefore FD = R.

Cor. 7. *In a right cone, the ſquare of the tranſverſe* $BA^2 = BE^2 - AE \times BH$, *the ſquare of the ſide of the fruſtum leſs the rectangle of the diameters of the baſes.* 110.

Let BF be perp. to the baſe AE, then $FE = \frac{BH + AE}{2}$; and (Geom. II. 23.) $AB^2 = AE^2 +$

$$BE^2 - 2AE \times \frac{BH + AE}{2} = BE^2 - AE \times BH.$$

Cor. 8. *In a right cone, the ſquare of the ſide of the fruſtum is equal to the ſum of the ſquares of the tranſverſe and conjugate,* $BE^2 = AB^2 + AE^2 + BH^2$.

Cor. 9. *In a right cone, the diſtance of the focus from the center, is equal to half the ſide of the fruſtum,* $\frac{1}{2}BE$.

For if S be the focus, O the center; C half the conjugate. Then (Prop. II.) $OS^2 = OB^2 + C^2 = \frac{1}{4}AB^2 + \frac{1}{4}AE \times BH =$ (Cor. 8.) $\frac{1}{4}BE^2$, and $OS = \frac{1}{2}BE$ or $\frac{1}{2}AH$.

Cor. 10. *In a right cone, if* F *be the focus,* C *the center, and* AG = AF. *The circle* GOI *parallel to the baſe of the cone* RS, *will cut the plane of the hyperbola* FAH *in the line* ED *the directrix of the hyperbola.* 111.

In the plane of the hyperbola, let FH, AL, be perp. to AF, and HD a tangent at H; then (XXVI.

Fig. and Sch.) D is the point thro' which the directrix
111. paſſes. But (XXV. Cor. 2) AL = AF = AG by conſtruction. Alſo (Cor. 9.) CF = AP = $\frac{1}{2}$ AN; and therefore CP drawn thro' the middle of AN, AM will be parallel to MN or to RS the baſe of the cone. Now (Prop. XV.) CF : CA : : CA : CD : : (by diviſion) AF : AD; that is, AP : CA : : AG : AD; therefore DG is parallel to CP, or the baſe of the cone. And on the contrary, if DG is parallel to RS, DE will be the directrix.

112. Cor. 11. *In a right cone, if* F *be the focus, and* AG = AF, *and the circle* GI *be drawn parallel to the baſe of the cone; then the diſtance of any point* P *from the focus as* FP, *will be equal to the diſtance of* P *from the periphery of the circle* GI.

Let PQ be an ordinate to AQ; draw QH, QS parallel to AG, DG. Then will (Geom. V. 12. Cor.) QH = SG = diſtance of P from the periphery of the circle GI; becauſe the plane SQP is parallel to the circle GI. By Cor. 10, DE is the directrix; therefore if DR be the focal tangent, and if the ordinate QP be continued to the tangent at R; then (Prop. XXV.) QR = FP, and AL = AF = AG. And by ſimilar triangles, AL : QR : : (DA : DQ : :) AG : QH; that is, AG : FP : : AG : SG; therefore FP = SG.

SCHOLIUM.

Hence we have the reaſon why the *Hyperbola* is called one of the *Conic Sections*. For if two oppoſite cones be cut by any plane, it will always make the figure of an hyperbola in both the cones, as is demonſtrated in this propoſition. The properties of which have been demonſtrated at large in this Book.

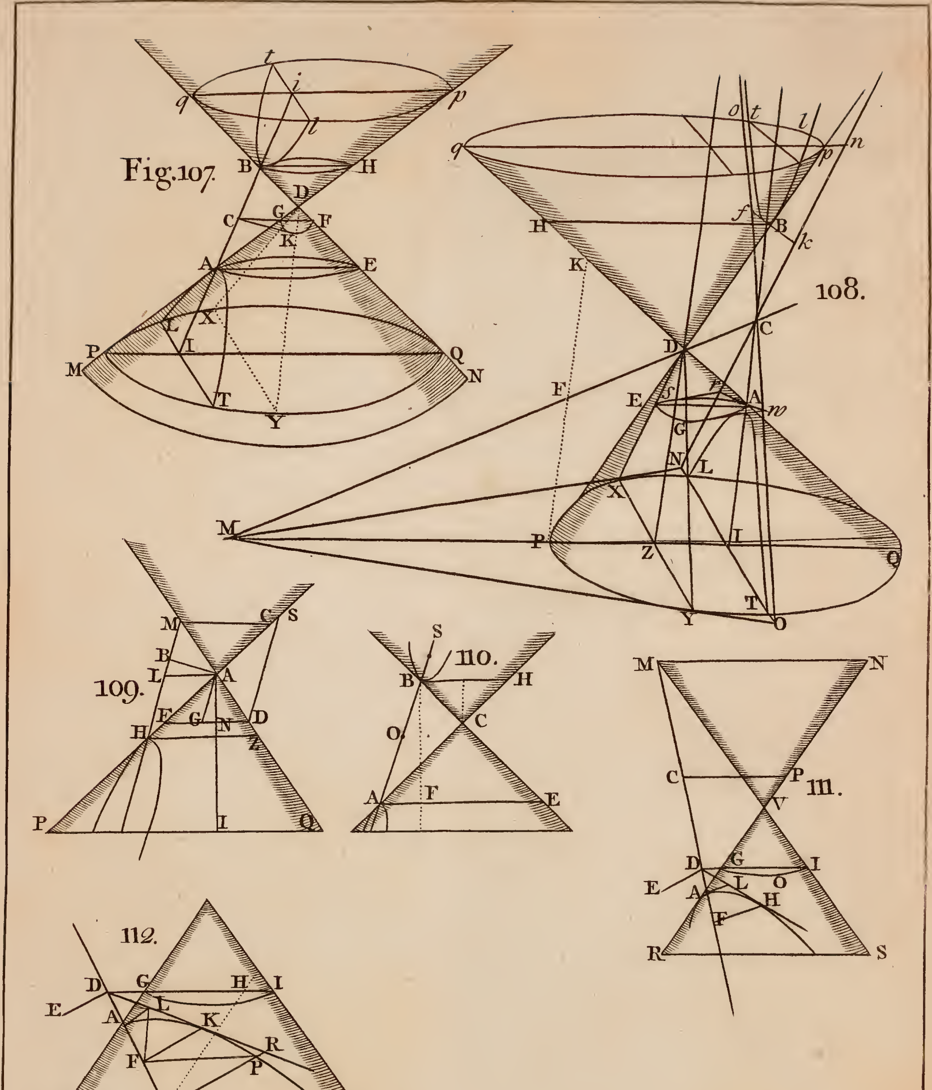


Fig.107.
t i q p l B H D C G F K A E L X P I Q M N T Y
108.
o t l q n p f H B k K C D F E s r A w G N L X M P Z I Q T Y O
109.
M C S B L A F G N D H Z P I Q
110.
S B H C O A F E
111.
M N C P V D G I E A L O H F R S
112.
D G H I E A L K F R P S Q

Pl. XXIV. *pa.164.*

CONIC SECTIONS.

BOOK III.

Of the PARABOLA.

DEFINITIONS.

DEFIN. I.

IF one end of a thread equal in length to CH, be fixed at the point F, and the other end fixed at H, the end of the ſquare DCH. And if the ſide CD of the ſquare be moved along the right line BD, and always coincides with it. Then if the ſtring FGH be always kept tight, and cloſe to the ſide GH of the ſquare. The point or pin G (where it leaves the ſquare) will deſcribe a curve MRALGK called a *Parabola*. Fig. 1.

DEF. II.

The fixed point F, is called the *Focus*.

DEF. III.

The right line BD, is called the *Directrix*.

DEF. IV.

If the line BN be drawn thro' the focus F, perpendicular to BD; then AN is called the *Axis* of the Parabola; and A the *Vertex*.

Fig. 1.

DEF. V.

A line drawn thro' the focus F, perpendicular to the axis, as LR is called the *Parameter* or *Latus Rectum.*

DEF. VI.

Any line drawn within the curve, parallel to the axis, as GH is called a *Diameter.* And the point G where it cuts the curve, is the *Vertex.*

DEF. VII.

2. A right line drawn from any diameter to the curve, and parallel to the tangent at the vertex, as PM, is called an *Ordinate.* If it go quite thro' the curve, it is called a *double Ordinate.*

DEF. VIII.

The part of the diameter between the vertex and ordinate, as GP is called the *Abſciſſa.*

DEF. IX.

A right line meeting the curve in one point G, but does not cut it, is called a *Tangent* in that point.

PROP. I.

3. *If* BD *be the directrix,* G *any point in the curve, the line* GD *drawn to the directrix, parallel to the axis, is equal to the line* GF *drawn from the ſame point* G *to the focus*; GD = GF.

For HG + GF = length of the ſtring = HD, take away GH from both, and then GD = GF.

Cor. 1. *The diſtances of the focus, and of the directrix from the vertex, are equal.* AB = AF.

For when D is at B, G will be at A; conſequently AB = AF.

Cor. 2. *If* GP *be an ordinate to the axis; then* $AP + AF = FG$. Fig. 3.

For $AP + AF = BP = GD$.

Cor. 3. $FG - FP =$ *half the latus rectum.*

PROP. II.

The diſtance of the focus from the vertex is $\frac{1}{4}$ *the latus rectum* : $AF = \frac{1}{4} LR = \frac{1}{2} LF$. 1.

For when the pin G comes to L, then $LF = FB =$ (Prop. I. Cor. 1.) $2FA$, and $AF = \frac{1}{2} FL$. For the ſame reaſon $FA = \frac{1}{2} FR$, therefore $FA = \frac{1}{4} LR$.

SCHOLIUM.

As the latus rectum to the axis is four times the diſtance of the vertex A from the focus F. So in any other diameter GH, four times the diſtance of its vertex from the focus, or AFG is called its *Latus Rectum.*

PROP. III.

The ſquare of any ordinate to the axis, is equal to the rectangle of the latus rectum and abſciſſa : $PM^2 = LR \times AP$. 4.

For (I. Cor. 2.) $MF = AF + AP =$ (Prop. II.) $AP + \frac{1}{4} LR$, and $FP = AP - AF = AP - \frac{1}{4} LR$. And in the right angled triangle MFP, $MP^2 = MF^2 - FP^2 = \overline{MF + FP} \times \overline{MF - FP} = 2AP \times \frac{1}{2} LR = AP \times LR$.

Cor. 1. *If* F *is the focus,* $MP^2 = AP \times 4AF$.

Cor. 2. *The abſciſſas are as the ſquares of their ordinates.*

$$AP : AQ :: PM^2 : QN^2.$$

Fig. 4. For $AP : AQ :: AP \times LR : AQ \times LR :: PM^2 : QN^2$.

Cor. 3. *The latus rectum is a third proportional to the abſciſſa and ordinate.* $AP : PM : LR \div$.

PROP. IV.

5. *As the latus rectum to the ſum of any two ordinates :: ſo their difference, to the difference of the abſciſſæ. Lat. rect.* $: CD :: ND : PQ$.

Let $L =$ latus rectum, then (Prop. III.) $L \times AP = PM^2$; and $L \times AQ = NQ^2$. And by ſubtraction, $L \times AQ - L \times AP = NQ^2 - PM^2$; therefore $L : NQ + PM :: NQ - PM : AQ - AP$; that is, $L : DC :: ND : PQ$.

Cor. 1. *If* MD *be the axis,* NC *an ordinate to it. Then the rectangle* $NDC = MD \times$ *parameter.*

Cor. 2. *The rectangle* NDC *is every where as* MD.

PROP. V.

6. *If a line* AN *be drawn from the vertex to cut the curve in* M, *and the ordinate* QO *in* N; *and the ordinate* PM *be drawn. Then the rectangle* $PM \times QN = AQ \times$ *parameter.*

Let $L =$ latus rectum, then (Prop. III) $QO^2 = L \times AQ$; and (III. Cor. 2.) $PM^2 : QO^2$ or $L \times AQ :: AP : AQ ::$ (ſimilar triangles) $PM : QN$; and $PM^2 \times QN = PM \times L \times AQ$, or $PM \times QN = L \times AQ$.

Cor. 1. $PM \times QN = QO^2$.

Cor. 2. *Any line beſides the axis, paſſing thro' the vertex* A, *will cut the curve again in ſome point* M.

For QN : QA : : latus rectum : PM the ordinate where it cuts the curve. Fig. 6.

P R O P. VI.

If from any point C *in the axis, the line* CF *be drawn, cutting the curve in* D *and* F, *and the ordinates* DB, FQ *be drawn. Then* $CA^2 = AB \times AQ$. 6.

For (III. Cor. 2) AB : AQ : : $BD^2 : QF^2$: : (ſimilar triangles) $\overline{CA + AB}^2 : \overline{CA + AQ}^2$; whence $AB \times \overline{CA^2 + 2CA \times AQ + AQ^2} = AQ \times \overline{CA^2 + 2CA \times AB + AB^2}$, and $AB \times \overline{CA^2 + AQ^2} = AQ \times \overline{CA^2 + AB^2}$; whence $\overline{AQ - AB} \times CA^2 = AQ^2 \times AB - AB^2 \times AQ = \overline{AQ - AB} \times AQ \times AB$. Therefore $CA^2 = AQ \times AB$.

P R O P. VII.

If AB *be perpen. to the axis* AE, EG *an ordinate, and* AG *drawn. Then any line* BF *being drawn parallel to the axis* AE, *cutting the curve in* M, *and* AG *in* E. *Then* BM, BD, BF *are continually proportional.* 7.

For (III. Cor. 2.) AP : AE : : PM^2 or $CD^2 : EG^2$: : (ſimilar triangles) $AC^2 : AE^2$; therefore AP : AC : : AC : AE; that is, BM : BD : : BD : BF.

P R O P. VIII.

If from any point M *in the curve,* MF *be drawn to the focus, and* MD *be drawn parallel to the axis cutting the directrix* BD *in* D; *and* DF *be drawn. Then the line* MG *drawn thro' the middle of* DF *will touch the curve in* M. 8.

From any other point *m* in the line MG, draw *m*F to the focus, and *md* perp. to BD. Then ſince

Fig. 8. MF = MD, and *m*F = *m*D, the triangles FMD and F*m*D are iſoceles, and FG = GD; but the angle D*dm* being right D*m* or *m*F is greater than *dm*; therefore the point *m*, by the generation of the parabola, is without the curve.

Cor. 1. *If the tangent* MD *be continued to cut the axis at* T; *the diſtance of the focus and interſection, is equal to the diſtance of the focus and point of contact.* FT = FM.

For MD being parallel to FT, the $<$ GTF = GMD = GMF; therefore FT = FM.

Cor. 2. *The tangent* MG *biſſects the angle* DMF *made by the line perp. to the directrix, and that drawn to the focus.*

Cor. 3. *A line drawn from any point* M *to the focus, and another parallel to the axis, make equal angles with the curve*; FM*m* = OM*r*.

PROP. IX.

9. *If the tangent at* M, *cut the axis* AB *at* T; *then the diſtance of the interſection from the vertex,* AT = AP, *the abſciſſa to the ordinate* PM.

From T draw any line TF cutting the curve in D and F, and draw the ordinates DB, FQ; then (Prop. VI.) $TA^2 = AB \times AQ$. Now let the points D, F, approach to M, and coincide with it; then B, Q coincide with P, and TDF becomes the tangent TM; in which caſe $TA^2 = AP \times AP = AP^2$, and TA = AP.

Cor. 1. *The ſubtangent* PT *is double the abſciſſa* AP.

Cor. 2. *The tangent at the vertex* A, *is perp. to the axis* AB.

For, from the generation, the ordinates are perp. to the axis, or parallel to the directrix. Fig. 9.

Cor. 3. *All parabolas having the ſame abſciſſa, will have their tangents meeting in the ſame point of the axis.*

For if AP be given, AT equal to it, will be given for them all.

PROP. X.

If two parabolas be deſcribed on the ſame axis; the ordinates will be proportional, or in a given ratio to one another. 10.

For let the common abſciſſa be AB, the ordinates BC, BD. l, L the parameters. Then (Prop. III.) $l \times AB = BC^2$, and $L \times AB = BD^2$; therefore $BC : BD :: \sqrt{l \times AB} : \sqrt{L \times AB} :: \sqrt{l} : \sqrt{L}$, which is a given ratio.

Cor. *The correſpondent ordinates of two parabolas deſcribed on the ſame axis, are in the ſubduplicate ratio of the parameters.*

PROP. XI.

If F *be the focus,* MO *perp. to the tangent* MT; *then the ſubnormal (or diſtance of the ordinate from the point of interſection)* $PO = \frac{1}{2}$ *the latus rectum.* 11.

Let L = latus rectum, then (IX. Cor. 1.) $PT = 2AP$; and by ſimilar triangles TP or $2AP : PM :: PM : PO$, and $2AP \times PO = PM^2 =$ (Prop. III.) $L \times AP$; therefore $2PO = L$ or $PO = \frac{1}{2}L$.

Cor. 1. *If* F *be the focus,* MO *perp. to the tangent* MT: *Then* $FT = FM = FO$.

For (Prop. II.) $AF = \frac{1}{4}L$, and (I. Cor. 2.) $FM = AP + AF = AP + \frac{1}{4}L$, and $FP = AP -$

AF =

Fig. AF = AP — $\frac{1}{4}$ L. And FO = AP — $\frac{1}{4}$ L + 11. $\frac{1}{2}$ L = AP + $\frac{1}{4}$ L = FM = (VIII. Cor. 1.) FT.

Cor. 2. *A circle deſcribed with the radius* FM, *and center* F, *will paſs thro'* T, M, O.

Cor. 3. *The angle* MFO *is double the angle* MTO.

Cor. 4. *If* OG *be perp. to* MF; *then* MG = *half the latus rectum.*

For ſince FM = FO, the < FMO = < FOM; therefore the right angled triangles MOP, MOG are ſimilar and equal; and MG = OP = $\frac{1}{2}$ L.

Cor. 5. *Hence* MG = OP = 2AF = $\frac{1}{2}$ *parameter.*

PROP. XII.

12. *If* MT *be a tangent at any point* M, AG *a tangent at the vertex*; *then if* FG *be perp. to* MT *from the focus, it will paſs thro' the interſection* G.

For if BD be the directrix, and MD parallel to the axis; then (Prop. VIII.) ſince MF = MD, MG drawn to the middle of FD is perp. to it, and FG = GD; and (I. Cor. 1.) ſince FA = AB; therefore AG is parallel to DB, or perp. to AF, and is therefore a tangent at F. Therefore the tangent AG, the tangent MT, and the perp. FG all interſect in one point G.

Cor. 1. *If* MP *be an ordinate,* AG *a tangent at* A, MT *a tangent at* M; *then* AG^2 = AP × $\frac{1}{4}$ *latus rectum* = AP × AF.

For (IX. Cor. 1.) ſince TA = $\frac{1}{2}$ TP, GA = $\frac{1}{2}$ MP; therefore GA^2 = $\frac{1}{4}$ MP^2 = (Prop. III.) AP × $\frac{1}{4}$ parameter.

Cor. 2. *If* MT, AG *be tangents*; *then* AG = *half the ordinate* MP.

Cor. 3. *If* FG *be perp. on the tangent* MT *from* Fig.
the focus. Then FA, FG, FM, *are in continual* 12.
proportion.

For in the right angled triangle FGT, FA : FG :: FG : FT or FM.

PROP. XIII.

If F *be the focus*, MO *perp. to the curve at* M. 11.
Then $MO^2 = FO \times$ *latus rectum.*

For (Prop. XI.) $PO = \frac{1}{2}$ parameter, and (XI. Cor. 1.) $MF = FO$. But (Geom. II. 23.) $MO^2 = MF^2 + FO^2 - 2FO \times FP = 2FO^2 - 2FO \times FP = 2FO \times \overline{FO - FP} = 2FO \times PO = FO \times$ parameter.

PROP. XIV.

If FH *be an ordinate at the focus*; THG *a tan-* 13.
gent at H, *or the focal tangent*; PM *an ordinate,*
continued to the tangent at G. *Then* $TP = FM = PG$.

Draw the tangent at the vertex AL. Then (IX.) $TA = AF =$ (Prop. I. Cor. 1.) $\frac{1}{4}$ parameter, and (def. V.) $FH = \frac{1}{2}$ parameter; therefore $TF = FH$, and conſequently by ſimilar triangles $TP = PG$; but $TP = TA + AP = AF + AP =$ (I. Cor. 2.) FM; and therefore $PG = FM$.

Cor. *If* TH *be the focal tangent*, AL *a tangent at the vertex*; *then* $TA = AL = AF = \frac{1}{4}$ *parameter*; *and* $TF = FH = \frac{1}{2}$ *parameter.*

Fig.

PROP. XV.

14. *If* AR *be the axis*, ML *any diameter*; AE, MT, *two tangents at* A *and* M. *Then the trictangle* AYT = MYE.

Draw the ordinate MP, then (IX. Cor. 1.) TA = $\frac{1}{2}$ TP; therefore AY = $\frac{1}{2}$ PM or $\frac{1}{2}$ EA = EY; therefore the triangles AYT, EYM are ſimilar and equal.

Cor. 1. *If* AD *be parallel to* MT; *the triangle* MPT = ADE = *rectangle* MPAE = DPAM = DATM.

This is evident by inſpecting the figure.

Cor. 2. *If* QIF *be perp. to* AB, *and* QRL *parallel to* MT; *then the triangle* QIR = *rectangle* IAEF.

For the triangles MPT, QIR are ſimilar; whence MPT : QIR : : MP^2 : QI^2 : : (III. Cor. 2.) PA : IA : : rectangle PAEM : rectangle IAEF; but MPT = MPAE; therefore QIR = IAEF.

Cor. 3. *The triangle* QLF = *parallelogram* LMTR.

For QIR = IAEF, add to both LRIF, then LQF = LRAE = LRTM.

Cor. 4. *After the ſame manner triangle* iqR = *rectangle* $i$AE$f$; *and* qfL = TRLM.

For the triangles MPT and qiR are ſimilar; whence MPT : qiR : : MP^2 : qi^2 : : AP : Ai : : PAEM : $i$AE$f$; but MPT = PAEM; therefore qiR = $i$AE$f$.

Then ſince TAY = MEY, add to both A$if$MY, and then T$if$M = A$if$E = iqR. Take ifLR from both; then TRLM = qfL.

Cor. 5. *Any ordinate* Qq *is biſſected by the diameter* ML.

For (Cor. 3. and 4.) triangle $LQF = LMTR$ Fig. $= Lqf$; therefore theſe triangles LQF, *Lqf*, are 14. ſimilar and equal, whence $LQ = Lq$.

PROP. XVI.

In any diameter MK, *the abſciſſas are as the ſquares* 15. *of the ordinates*; $ML : MK :: LQ^2 : KG^2$.

For draw the tangent MT to cut the axis AS in T. Then (XV. Cor. 3.) triangle $QFL =$ parallelogram LMTR, and $GBK = KSTM$. But $QL^2 : GK^2 ::$ triangle $QLF : GKB ::$ Par. $LMTR : KMTS :: LM : KM$.

Cor. *Any diameter interſects the curve only in one point, viz. the vertex.*

For the abſciſſas and ordinates continually increaſing, the curve recedes more and more from the diameter.

PROP. XVII.

In any diameter ML, *the ſquare of any ordinate,* 16. *is equal to the rectangle of the parameter and abſciſſa*; $LQ^2 = 4FM \times ML$.

From M draw the tangent MT cutting the axis in T, and draw AK parallel to it, from the principal vertex, and draw the ordinate MP to the axis. Then (III. Cor. 1.) $4AF \times AP = PM^2$, and $MK = AT =$ (IX.) $AP = \frac{1}{2}PT$. In the right angled triangle TPM, MT^2 or $AK^2 = MP^2 + PT^2 = 4AF \times AP + 4AP^2 = 4AP \times \overline{AF + AP} =$ (II. Schol.) $4AP \times FM = 4MK \times FM$; that is, $AK^2 = MK \times 4FM$.

But (XVI.) $AK^2 : QL^2 :: MK : ML :: 4FM \times MK : 4FM \times ML$, but $AK^2 = 4FM \times MK$; therefore $QL^2 = 4FM \times ML$.

Cor.

Fig. 16. Cor. 1. *The parameter is a third proportional to the abſciſſa and ordinate.*

Cor. 2. *If* MP *be perp. to the axis* AP, *then the latus rectum of the diameter* ML, *exceeds the latus rectum of the axis, by* 4AP.

For 4FM = (I. Cor. 2.) 4AF + 4AP, and 4AF is the latus rectum of the axis.

Cor. 3. *If* MC *is perp. to the tangent* MT, *then* TC *is half the latus rectum of the diameter* ML.

For TP : TM : TC ∺, or 2MK : AK : $\frac{1}{2}$ parameter.

3. Cor. 4. *The line* GD, *perp. to the directrix* BD, *is* $\frac{1}{4}$ *of the latus rectum of the diameter* GH.

By Prop. I.

SCHOLIUM.

Since the ſquare of the ordinate is always equal to the rectangle of the abſciſſa and latus rectum; for that reaſon the curve is called a *Parabola.*

PROP. XVIII.

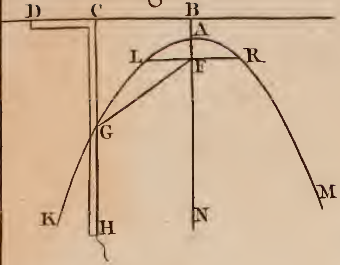
17. *In any diameter* DP, *the whole ordinate paſſing thro' the focus* F, *is the latus rectum to that diameter,* MN = *latus rectum of* DP; *and its abſciſſa* DP *is* $\frac{1}{4}$ *the latus rectum.*

Draw the tangent DT, then DP = TF = (VIII. Cor. 1.) DF = (II. Schol.) $\frac{1}{4}$ lat. rectum = $\frac{1}{4}$ L. But (XVII.) NP^2 or $PM^2 = L \times DP = 4DP \times DP = 4DP^2$; therefore PM = 2DP = $\frac{1}{2}$ L, and NM = L.

Cor. 1. *If an ordinate of any diameter paſs thro' the focus, as* MN. *The abſciſſa is equal to the diſtance of the focus from the vertex:* DP = DF.

Cor.

Fig.1.



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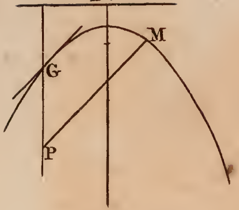

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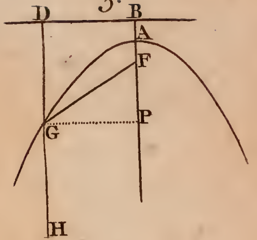

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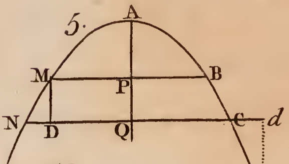

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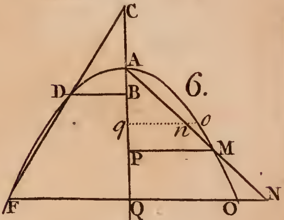

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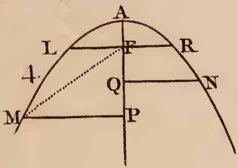

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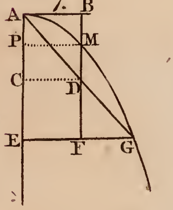

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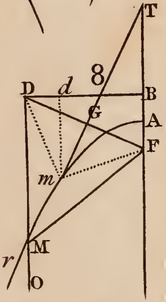

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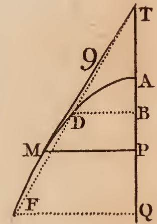

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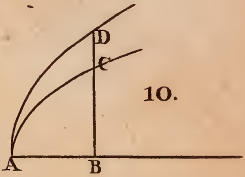

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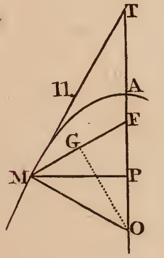

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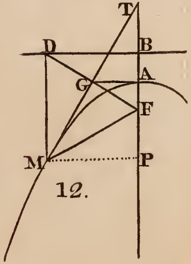

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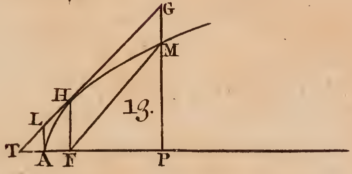

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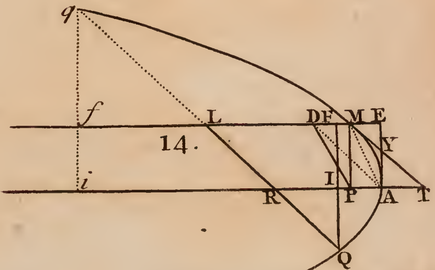

Pl.XXV. pa.176.

Cor. 2. *If any line* MN *be drawn thro' the focus* Fig.
F, *the rectangle* $NFM = \frac{1}{2} NM \times \frac{1}{2}$ *parameter.* 17.

For $NP^2 = DP \times NM$, becauſe NM is the latus rectum; and DT^2 or $PF^2 = DO^2 + TO^2 = 4AF \times AO + 4AO^2 = AO \times \overline{4AF + 4AO} =$ (becauſe $AO = TA$, and $4AF + 4AO =$ lat. rectum of DP) $TA \times MN$. But rectangle $NFM = NP^2 - PF^2 = DP \times NM - TA \times NM = AF \times NM$.

PROP. XIX.

If ML *be any diameter,* HP *an ordinate to it;* 18.
and HL *perp. to it. Then* $HL^2 = MP \times$ *latus rectum of the axis.*

Let BO be the directrix, continue PM to O, draw the tangents MT, AD, alſo draw FDO from the focus F. Then (VIII. Cor. 1.) $TF = FM$, and FD perp. to MT; alſo (XII.) AD perp. to FT. Then (XVII.) $HP^2 = 4MF \times MP$. The triangles HPL, DTA, DTF are ſimilar; whence $HP^2 : HL^2 :: DT^2 : DA^2 :: DF^2 : FA^2 ::$ TF or $MF : FA :: 4MF \times MP : 4AF \times MP$; but $HP^2 = 4MF \times MP$; therefore $HL^2 = 4AF \times MP$.

Cor. 1. *The ſquares of the perpendiculars* HL, *are as the abſciſſas* MP; *or* $HL^2 : hl^2 :: MP : Mp$.

Cor. 2. *If from the vertices* M, H, *of two dia-* 19.
meters MP, HG; *two ordinates be drawn* MN *and* HP; *then the abſciſſas* MP, HN, *will be equal.*

For if L = parameter; then $MG^2 = L \times HN$, and $HL^2 = L \times MP$; but $MG = HL$; therefore $L \times HN = L \times MP$, and $MP = HN$.

PROP.

Fig.

PROP. XX.

20. *If from any point* M *a tangent be drawn to cut any diameter* AP *in* T; *and the ordinate* MP *be drawn*; *then the diſtance of the interſection from the vertex*, AT; *is equal to the abſciſſa* AP.

Draw the diameter MN, and draw MG, AL, perp. to AP, MN; and draw AN parallel to MT, and AN will be an ordinate to MN. Then (XIX. Cor. 2.) AP = MN = (by parallel lines) AT.

Cor. 1. *The ſubtangent* PT *is double the abſciſſa* AP.

Cor. 2. *The tangents from each end of the double ordinate* M*m*, *meet in the ſame point of the diameter* T.

For the abſciſſa of the diameter drawn thro' *m*, will be equal to AP = AT.

14. Cor. 3. *Hence if* AR, ML *are any diameters beſides the axis*; *and* AE, MT, *tangents*; *the triangle* AYT = MYE.

For (this Prop.) ME = DM = AT; and the triangles AYT, MYE are ſimilar, and therefore equal.

PROP. XXI.

21. *If* BD *be the directrix*, GL *any diameter*, ML *an ordinate*, MT *a tangent*, MK *perp. to* GL. *Then the ſquare of the tangent* MT^2, *is to the ſquare of the ordinate*, ML^2 : : *as the diſtance of the point of contact from the directrix*, MD; *to* $\frac{1}{4}$ *the latus rectum of* GL.

For (Geom. II. 22.) $MT^2 = ML^2 + TL^2 + 2TLK$. But (XVII.) $ML^2 = 4FG \times GL$, and (XX.) $TL^2 = 4GL^2$, and $2TLK = 4GL \times LK$; whence $MT^2 = 4FG \times GL + 4GL^2 + 4GL \times LK$

$\times LK = 4GL \times \overline{FG + GL + LK} = 4GL$ Fig.
$\times FK$. Therefore $MT^2 : ML^2 :: 4GL \times FK :$ 21.
$4FG \times GL :: FK : FG$.

Cor. 1. *If the diameter* GL *becomes the axis* AC; 16.
then the ſquare of the tangent MT^2, *to the ſquare of the ordinate* $MP^2 :: AF + AP$, *to* AF; *that is, as latus rectum of* MK : *to latus rectum of the axis.*

For if F be the focus, AF (fig. 16.) = AB (fig. 21.)

Cor. 2. *The ſquare of the tangent* MT^2, *is to the* 21.
ſquare of the perp. MK^2 :: *as the diſtance of the point of contact and directrix* MD, *to* $\frac{1}{4}$ *parameter of the axis* AB.

For (XIX.) $MK^2 = 4AB \times GL$. And $MT^2 : ML^2 :: FK : FG$, and $ML^2 : MK^2 :: 4FG \times GL : 4AB \times GL :: FG : AB$. And ex equo, $MT^2 : MK^2 :: FK : AB$.

Cor. 3. $ML^2 : MK^2 :: FG : AB$.

For $ML^2 : MK^2 :: 4FG \times GL : 4AB \times GL :: FG : AB$.

Cor. 4. $LK^2 = 4HG \times GL$.

For $LK^2 = ML^2 - MK^2 = 4FG \times GL - 4AB \times GL = 4HG \times GL$.

Cor. 5. $MT^2 =$ *rectangle* $4FK \times GL$.

PROP. XXII.

If AG *be any diameter*, AB *a tangent*, AF *perp.* 22.
to the axis. BD *parallel to* AG.

Then $AF^2 = BD \times$ *latus rectum of the axis.*

For let L = latus rectum of AG, l = latus rectum of the axis. Draw the ordinate GD. Then (XVII.) $L \times AG = GD^2$; that is, $L \times BD = AB^2$. But (XXI. Cor. 1.) $AB^2 : AF^2 :: L : l ::$

Fig. $L \times BD : l \times BD$. But $L \times BD = AB^2$; therefore
22. $l \times BD = AF^2$.

Cor. 1. *The line* BD *is every where as the ſquare of the baſe* AF.

Cor. 2. $AF : BD :: FH : FD$.

For $l \times BD = AF^2$, and (IV. Cor. 1.) $AF \times FH = l \times DF$; whence $AF : BD :: l : AF :: FH : DF$.

PROP. XXIII.

23. *If* FG, HQ *be ordinates to the axis* AQ; BFT *a tangent at* F; FL *parallel to* AQ; *and* QH *produced to the tangent at* B. *Then* $BH : HL :: HL : FM$.

Put p for the parameter, then ſince $TG = 2AG$, by ſimilar triangles, $2AG : GF :: FL : LB :: p \times LF : p \times LB$; then $p \times LF \times GF = p \times 2AG \times LB =$ (III.) $2FG^2 \times LB$, and $2FG \times LB = p \times LF =$ (IV. Cor. 1.) $HL \times LC$, which reduced to a proportion, $BL : LH :: LC : 2FG$ or FM; and by diviſion, $BL - LH : LH :: LC - FM : FM$; that is, $BH : LH :: KC$ or $LH : FM$.

Cor. 1. $FM \times LB = p \times FL = HL \times LC$.

Cor. 2. *If* BF *be a tangent at* F, FL *a diameter*, BLC *perp. to* FL; *then* $BL^2 =$ *rectangle* HBC.

For ſince $BH : HL :: HL : FM$. By compounding $BH : (BH + HL)\ BL :: HL : (HL + FM)\ HK$. And by addition, $BH : BL :: (BH + HL)\ BL : (BL + HK)\ BC$.

PROP.

PROP. XXIV.

If AD, PQ, *be two diameters*; AC *a tangent at* A *interſecting* QP *in* C; *and if any point* F *be taken in* AD, *and* CL *be made equal to* AF; *and the lines* PF, AL, *drawn*; *they will interſect at* S, *a point in the curve.* 24.

Draw BS parallel to AD, and the ordinates GS, DP. Then ſince AF = CL, PL + AF = PC = AD. By ſimilar triangles, AF : PL : : AS : SL : : AB : BC; and compounding AF : AF + PL or AD : : AB : AB + BC or AC. Again by ſimilar triangles BS : CL or AF : : AB : AC. And multiplying, $BS : AD :: AB^2 : AC^2$; that is, $AG : AD :: GS^2 : DP^2$. Therefore (XVI.) as P is in the curve, S is in the curve alſo.

Cor. *Lines drawn from* A *and* P *thro' any point* S *in the curve, will cut off* AF *in the diameter* AD, *equal to* CL *the part from the tangent* AC.

PROP. XXV.

If any diameter interſect two parallel lines; *the rectangle of the ſegments of the lines, will be as the ſegments of the diameter.* DBE : FCG : : AB : AC. 25.

Draw the diameter PL of the parallels DE, FG, which will be parallel to AC; and draw the ordinate AR. Then the rectangle $DBE = HE^2 - HB^2 = HE^2 - RA^2 =$ (XVII.) $L \times PH - L \times PR$, L beingthe latus rectum of PL. Likewiſe rectangle $FCG = LG^2 - LC^2 = L \times PL - L \times PR$. Therefore $DBE : FCG :: L \times \overline{PH - PR} : L \times \overline{PL - PR} :: RH : RL :: AB : AC$.

Fig. 25. Cor. 1. *If* DE *be an ordinate to any diameter* PL, *and* DE *be cut by any other diameter* AC. *Then the rectangle* DBE = AB × *parameter of* PL.

Cor. 2. *If* PQ *be a tangent at* P; *then rectangle* FCG : PQ^2 : : AC : AQ.

For then D and E coincide with P, and HE vaniſhes.

26. Cor. 3. *If any line* FG *be cut by two diameters* AC, NO; *the rectangles of the parts, are as the ſegments of theſe diameters.* FCG : FOG : : AC : NO.

For (Cor. 1) FCG = AC × L; and FOG = NO × L; therefore FCG : FOG : : AC × L : NO × L : : AC : NO.

Cor. 4. *If two parallel lines*, DE, FG, *be cut by two diameters* AB, NO; *the rectangles of the parts will be as the ſegments of the reſpective diameters*; DBE : FOG : : AB : NO.

For DBE : FCG : : AB : AC; and FCG : FOG : : AC : NO. Therefore ex equo, DBE : FOG : : AB : NO.

Cor. 5. *And if* PQ *be a tangent at* P; *then the rectangle* FOG : PQ^2 : : NO : AQ.

PROP. XXVI.

27. *If two parallel lines* DE, FG, *interſect any other two parallels*, HL, MN, *in* B *and* O. *The rectangle* DBE : HBL : : FOG : MON.

Thro' the interſections B, O, draw the diameters AB, VO. Then (XXV. Cor. 4.) DBE : FOG : : AB : VO : : HBL : MON.

Cor. 1. *If any line* FG *cut two parallels* HL, MN, *in* O *and* I; *then rectangle* HIL : MON : : FIG : FOG.

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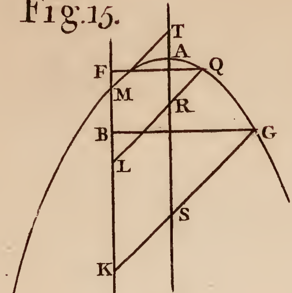
Fig. 15.



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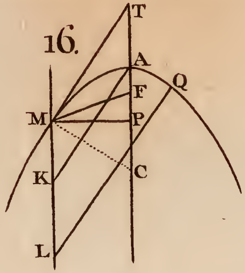

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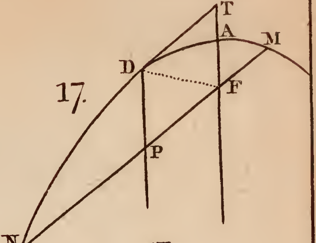

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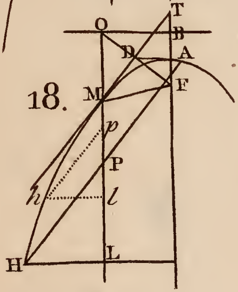

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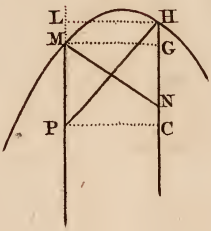

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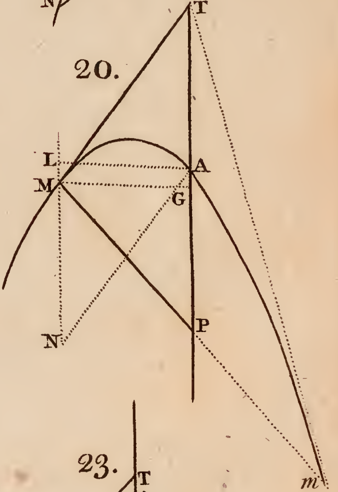

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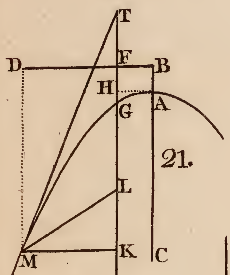

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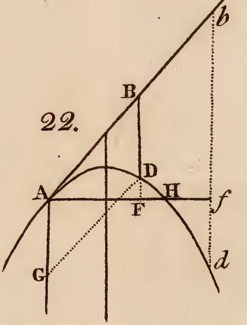

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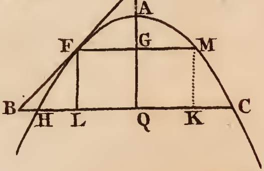

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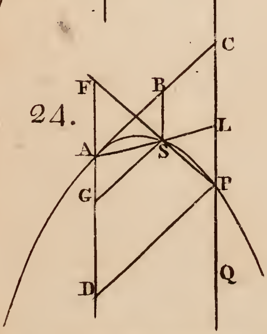

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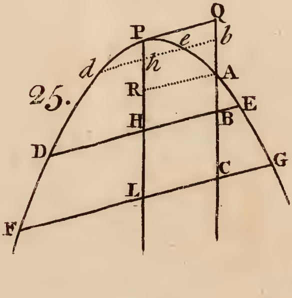

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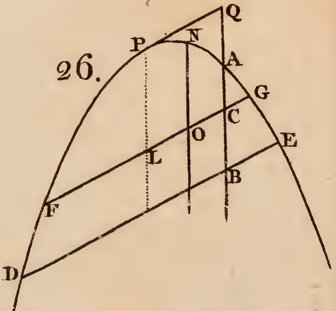


Pl. XXVI. *pa. 182.*

This is plain by ſuppoſing DE to fall upon FG.

Fig. 28. Cor. 2. *If* BD, BL *be tangents at* D *and* L, *and* KF, RN, *parallel to them. Then* GOF : ION : : DR^2 : NRI : : BD^2 : BL^2 : : GKF : KL^2 : : *parameter of the diameter paſſing thro'* D : *parameter of the diameter thro'* L.

For if DE, HL (fig. 27.) become tangents; then D and E coincide, as alſo H and L; and DBE, HBL become BD^2 and BL^2; and the like for the reſt.

Alſo BD, and BL are equal to the ordinates drawn to the diameters at D and L, from the point where the diameter from B cuts the curve. And if *d* and *l* be the parameters, H the common abſciſſa, then $BD^2 : BL^2 : d \times H : l \times H : : d : l$.

PROP. XXVII.

29. *If* DM, DN *be two tangents,* MN *the line joining the points of contact; and if any line* GL *be drawn parallel to either tangent* DN, *cutting the curve in* R *and* L, *and* MN *in* H. *Then* GR, GH, GL *are continually proportional.*

For (XXVI. Cor. 3.) RGL : GM^2 : : DN^2 : DM^2 : : (ſimilar triangles) GH^2 : GM^2. Therefore RGL $=$ GH^2.

PROP. XXVIII.

30. *If* HT, KT, *be two tangents,* HK *the line drawn thro' the points of contact. Then the line drawn from the interſection* T, *thro'* O *the middle of* HK, *will biſſect all the lines within the curve, parallel to* HK : *viz. if* HO $=$ OK, *then* IP $=$ PN.

By Cor. 2. Prop. XX. The tangents HT, KT meet in the diameter belonging to the ordinate

Fig. 30. HK. Therefore ſince HO = OK, TO is the diameter. And ſince IN is parallel to HK, IP and PN are ordinates, and IP = PN, by Cor. 5. Prop. XV.

Cor. 1. *If* LM *be drawn parallel to* HK *joining the points of contact; the parts intercepted between the curve and tangent on each ſide are equal,* LI = NM.

For ſince HO = OK, by ſimilar triangles LP = PM; take away the equals IP = PN; then the remainder LI = MN.

Cor. 2. *The tangent at* A *the vertex, is biſſected in that point.*

Cor. 3. HO, OK, *and alſo* IP, PN, *are ordinates to the diameter paſſing thro' the interſection* T *of the tangents.*

PROP. XXIX.

31. *If* TH, TK *be two tangents, and* HK *the line drawn thro' the points of contact. And from the interſection* T, *any line* TI *be drawn cutting* HK *in* D, *and the curve in* F *and* I. *Then* TF : TI : : DF : DI.

Thro' F and I draw AC, LM parallel to HK; then (XXVIII. Cor. 1.) LI = NM, and AF = BC, whence MI = LN, and CF = AB.

By ſimilar triangles AF : LI : : TF : TI,
and CF : MI : : TF : TI.
and multiplying, $AF \times CF : LI \times MI :: TF^2 : TI^2$.
that is, $AF \times AB : LI \times LN :: TF^2 : TI^2$.

But (XXVI. Cor. 2.) $AF \times AB : LI \times LN :: HA^2 : HL^2 ::$ (parallel lines) $DF^2 : DI^2$. Therefore $TF^2 : TI^2 :: DF^2 : DI^2$. And TF : TI : : DF : DI.

Cor. *If it be* TF : TI : : DF : DI. *The line* HK Fig.
which joins the points of contact, will paſs thro' the 31.
point D.

PROP. XXX.

If AF, AG, *be two tangents*; FG *a line drawn thro' the points of contact; and thro' the interſection of the tangents,* AV *be drawn parallel to* GF. *Then from any point* V, *if there be drawn* VL *thro'* O *the middle of* FG, *to cut the curve in* P *and* L. *Then will* VP : VL : : OP : OL. 32.

Draw AQRL, and thro' Q draw PSQ parallel to FG. Then (XXIX.) AQ : AL : : RQ : RL, whence AL × RQ = RL × AQ.

By ſimilar triangles, PQ : OR : : LQ : LR; and OR : SQ : : AR : AQ. Whence by compounding PQ : SQ : : LQ × AR : LR × AQ. And by diviſion PS : SQ : : LQ × AR — LR × AQ : LR × AQ.

But LQ × AR — LR × AQ = $\overline{LR + QR}$ × AR — LR × AQ = QR × AR + LR × QR = AL × QR = (before) RL × AQ; therefore PS = SQ. And conſequently as Q is in the curve, P is in the curve. And by reaſon of the parallels, OR, PQ, VA; VP : VL : : AQ : AL : : (XXIX.) QR : RL : : OP : OL.

PROP. XXXI.

Let AO *be a diameter*; FA, GA, *two tangents*; FG *an ordinate. Let two other tangents* VH, VI, *be drawn from any point* V, *in the line* VA *being parallel to* FG. *I ſay* HI *drawn thro' the points of contact, will paſs thro' the point* O, *the middle of* FG. 33

For thro' O and V draw VPOL cutting the curve in P and L. Then ſince VA is parallel to

Fig. the ordinate FG; therefore (XXX.) it is, VP : 33. VL : : OP : OL. And ſince the tangents VH, VI meet in V; therefore (XXIX.) VP : VL : : OP : OL. Therefore the point O is common to both FG and HI; and ſo HI paſſes thro' O.

Cor. 1. *If* A, V, *be the concourſe of the tangents* FA, GA, *and* HV, IV; *then the line, (joining the points of contact of any two tangents meeting in* AV) *will paſs thro'* O.

Cor. 2. *If* FO, GO, *be ordinates to the diameter* AO; *and any line* HI *be drawn thro'* O; *then the tangents* HV, IV, *will meet ſomewhere in the line* AV *drawn parallel to* FG.

PROP. XXXII.

34. *If* VH, VI, *be two tangents at* H *and* I; *and* IHD *be drawn thro' the points of contact. And if any line* VPOL *be drawn thro'* V; *and the tangents* PD, LD *be drawn; they will meet ſomewhere in the line* ID.

Draw FG thro' O, where VL and IH interſect; ſo that GO may be = OF; and draw VAD parallel to FG. Then (XXXI. Cor. 2.) the tangents PD, LD, will meet ſomewhere in the line VA, as at D. Then if DO does not paſs thro' the points of contact H, I; let a line drawn from D thro' N, paſs thro' the points of contact. Then (XXIX.) we ſhall have VP : VL : : NP : NL. But HV, IV, being tangents, we have VP : VL : : OP : OL. Therefore the former proportion cannot be true, except N and O coincide. Therefore DO paſſes thro' the points of contact H, and I. That is, the tangents PD, LD, meet in the line IH.

Cor. *If* HV, IV *be two tangents, and* IHD *be drawn thro' the points of contact*, I, H. *And from any*

any point D, *the tangents* DP, DL *are drawn*; *then* Fig.
LP *drawn thro' the points of contact* L, P, *will paſs* 34.
thro' V.

For a line drawn thro' V will paſs thro' the points of contact L, P, and therefore no other can.

PROP. XXXIII.

If MD, ND *be two tangents at* M *and* N; *and* 35.
MF, NF *be drawn to the focus* F. *The angle* MFN
= *twice the angle* MDN.

Draw the axis TAFB. Produce MD to cut the axis in T; and let ND cut it in G. Then (XI. Cor. 3.) the $< MFB = 2 < MTB$, and $< NFB = 2 < NGB$; therefore $MFB + BFN$, that is, $MFN = 2MTG + 2NGA = 2DTG + 2DGT = 2MDG$ (Geom. II. 1.)

Cor. 1. *If* MD, ND, *be two tangents, and the* 36.
line MN *joining the points of contact, paſs thro' the focus* F. *Then* MDN *is a right angle.*

For when MFN is a right line, the $< MFB + BFN$ is equal to two right angles; and therefore MDN is one.

Cor. 2. *If* MD, ND *be two tangents, and the line* MN *joining the points of contact, paſſes thro' the focus* F. *Then* MD, ND, *meet ſome where in the directrix* GE.

For the focal tangent being drawn will cut the axis in G, ſo that AF = AG (by Prop. IX.), which is the point through which the directrix is drawn (I. Cor. 1.), and (XXXI. Cor. 2.) the tangents MD, ND will meet ſome where in GE.

Cor. 3. *If* MD, ND *be two tangents, and* MN *paſſes thro' the focus* F. *Then the line* DF, *drawn from the interſection of the tangents to the focus, is perp. to* MN.

For

Fig. 36. For if GE be the directrix; and ME parallel to the axis. Then (Prop. I.) $ME = MF$, and (VIII. Cor. 2.) $\angle DME = DMF$; therefore the triangles MDE, MDF are ſimilar and equal, for MD is common to both, whence $ED = DF$, and $\angle MFD = MED =$ a right angle.

PROP. XXXIV.

37. *If* AB *be any diameter, and any line* AP *be drawn from* A, *thro' any point* P *of the curve; and thro'* P, *the line* NP *be drawn parallel to* AB. *Then if the ordinate* DF *be drawn to interſect theſe lines in* E *and* G; DE, DF, DG *will be continually proportional.*

Draw the ordinate PO; then by ſimilar triangles, $OP : DE :: AO : AD ::$ (XVI.) $OP^2 : DF^2$, and $OP \times DF^2 = DE \times OP^2$, or $DF^2 = DE \times OP$ or $DG \times DE$.

Cor. *As* $AD : DE :: OP :$ *latus rectum of* AB. For $DE \times OP = DF^2 = AD \times$ parameter.

PROP. XXXV.

38. *If any line* MN *cut the directrix* TO *in* O, *and from the points* M, N, *where it cuts the curve,* MF, NF *be drawn to the focus* F; *and* MF *be produced to* H. *Then* OF *drawn to the focus, will biſſect the angle* NFH.

Draw MP, NQ perp. to TO; and ND parallel to MF. Then by ſimilar triangles, $MP : NQ :: MO : NO :: MF : ND$. But (Prop. I.) $MP = MF$; therefore $ND = NQ = NF$. Whence the angle $NFD = NDF = DFH$, by the parallels ND, FH.

Cor. *Hence if the angle* NFH *be biſſected by the* Fig.
line FO *drawn from the focus; it will cut* MN *at* 38.
O *in the directrix.*

PROP. XXXVI.

If QT, PT *be two tangents. And if* QF, PF 39.
be drawn from the points of contact to the focus.
Then the line TF *will biſſect the angle* PFQ.

Let DR be the directrix; draw QPR, and produce TF to K; and draw LR, KR, tangents at L and K (XXXII. and Cor.); let QD, PE, be perp. to MR; then by ſimilar triangles, QD : PE : : QR : PR : : (XXIX.) QN : PN. That is, (Prop. I.) QF : PF : : QN : PN. Therefore (Geom. II. 25. Cor. 1.) FN biſſects the angle QFP.

PROP. XXXVII.

If a trapezium ABDC *be inſcribed in a parabola,* 40.
and any point E *be taken in the curve, from which*
two lines EN, EH, *are drawn parallel to any two*
adjoining ſides, as AB, AC, *to interſect the oppoſite*
ſides in N, Q, *and* H, R. *Then taking the rectan-*
gles on the oppoſite ſides; it will always be ER ×
EH *to* EN × EQ, *in a given ratio.*

Draw DFG, BLP, parallel to AC; thro' L and C, draw COLS. Thro' the middle of AC, BL, draw the diameter MK, which will biſſect the parallels DF, TE; and alſo SG, OR, terminated at AB, CL. Then (XXVIII. Cor. 1.) will SD = FG, and OT = ER, which would alſo be ſo, if SG, RH were tangents.

By ſimilar triangles and parallel lines, HO : SD : : OC : SC : : AR : AG. And alternately, HO : AR or EQ : : SD or FG : AG. Again, BP or

ER :

Fig. ER : PN : : DG : GB. And multiplying, HO ×
40. ER : EQ × PN : : FG × DG : AG × GB : : (Prop. XXVI.) ER × RT or ER × EO : RA × RB or EQ × EP. And by compoſition, FG × DG : AG × GB : : HO × ER + EO × ER : EQ × PN + EQ × EP : : EH × ER : EQ × EN. And it is evident, that as long as the points A, B, D, C, are fixed; that FGD to AGB is a given ratio, and therefore that of REH to QEN.

Cor. 1. *The ſame things ſuppoſed; it will be*, EH : EN : : EO : EP : : HO : PN.

For EH × ER : EQ × EN : : FG × GD : AG × GB : HO × ER : EQ × PN. And dividing all the terms, EH : EN : : HO : PN. And dividing, EH : EN : : EO : EP.

Cor. 2. *If the points*, A, B, C, E, *are fixed, and* D *variable at pleaſure; it will ſtill be* EH *to* EN, *and* HO *to* PN, *in the given ratio of* EO *to* EP *or of* RT *to* RB.

Cor. 3. *If the points* A, B, *coincide*; AG *then becomes a tangent. And if* C *and* D *alſo coincide*; DH *becomes a tangent. And then* Q, N, *coincide.*

41. Cor. 4. *If* BC *be drawn to cut* EH *in* I, *and if it be* EH : EI : : EN : EV. *Then if* BV *be drawn it will touch the curve in* B.

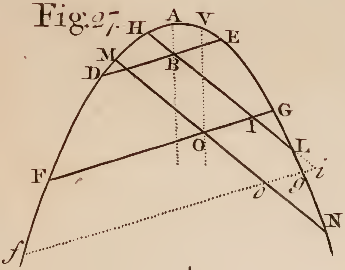
For (Cor. 2.) if D be moveable, it will be EH : EN : : EO : EP, wherever D is. Therefore ſuppoſe D to move round till it coincide with B; then BND will become a tangent at B, and N will fall upon V; that is, BV will be a tangent.

40. Cor. 5. *If you make* EH : EN : : EO : EP, *and draw* OC, BP, *they will meet at* L *in the curve.*

For ſuppoſing L a point in the curve, it will be by (Cor. 2.) EH : EN : : EO : EP; therefore L cannot be out of the curve.

Cor.

Fig. 27.



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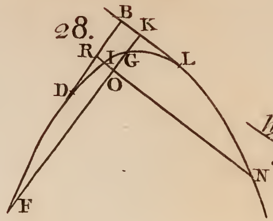

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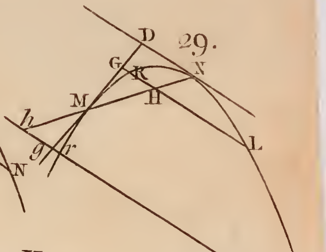

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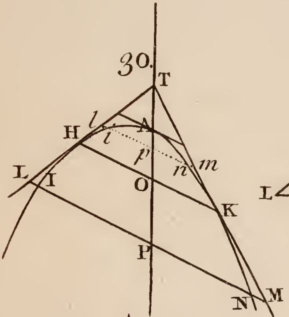

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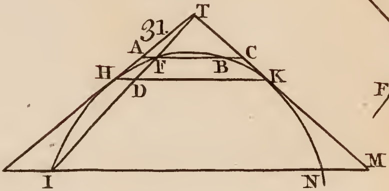

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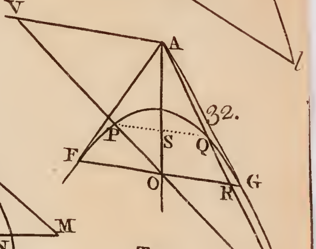

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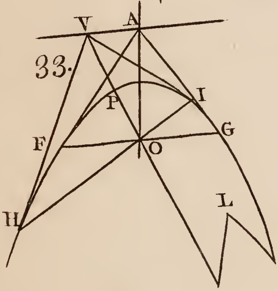

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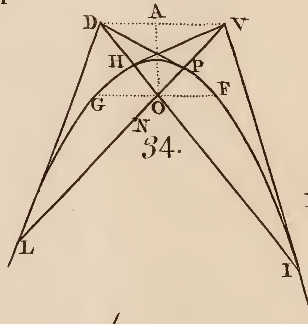

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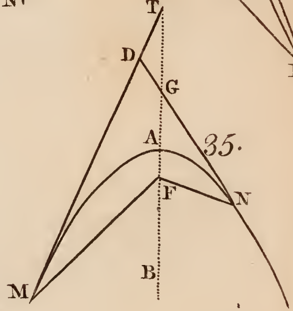

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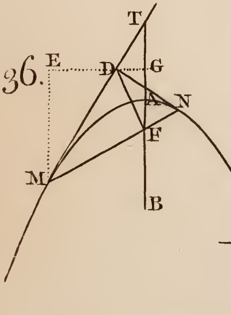

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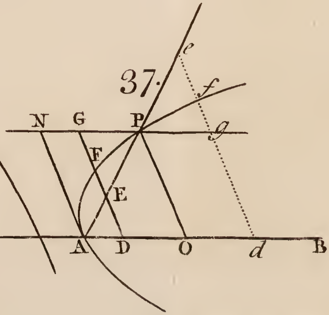

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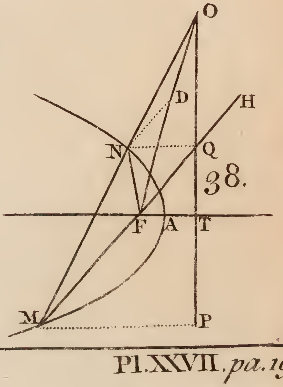


Cor. 6. *If there be five points* B, L, D, C, E; Fig.
and thro' any point E, *there be drawn* EH, EN, *cutting* BD, BL, DC, LC, *in* N, P, H, O, *ſo that* 40.
EH : EN : : EO : EP. *Then* CA, BA *being drawn parallel to* EH, EN; *they will meet at* A *in the curve.*

This is the reverſe of the foregoing Cor. For if A was out of the curve, L would not be in it.

Cor. 7. *If* A *and* B *coincide, then* AB *becomes a tangent. And the like for any other two points.*

PROP. XXXVIII.

If a trapezium ABDC *be inſcribed in a parabola, and from any point* E *of the curve, four lines* EL, EK, EO, EM, *be drawn to the four ſides of the trapezium, in any angles whatever, equal or unequal, and cutting them in* L, K, O, M. *And likewiſe from any other point e, four more lines be alſo drawn to the ſides of the trapezium, reſpectively parallel to the former, cutting them in l, k, o, m. Then taking the rectangles of the oppoſite ſides; the rectangle* EL × EM : *el* × *em* : : EK × EO : *ek* × *eo*. 42.

Thro' E, *e*, draw QEN, *qen*, parallel to AB; and EHR, *ehr*, parallel to AC. Then by ſimilar triangles, ER : *er* : : EL : *el*; and EH : *eh* : : EM : *em*. And by multiplying ER × EH : *er* × *eh* : : EL × EM : *el* × *em*.

Again, EQ : *eq* : : EK : *ek*, and EN : *en* : : EO : *eo*. And multiplying, EQ × EN : *eq* × *en* : : EK × EO : *ek* × *eo*. But (Prop. XXXVII.) ER × EH : *er* × *eh* : : EQ × EN : *eq* × *en*. Therefore EL × EM : *el* × *em* : : (EQ × EN : *eq* × *en* : :) EK × EO : *ek* × *eo*.

Cor. 1. *If* ABCD *be an inſcribed trapezium; and if* AB, CD, *produced interſect in* X; *and the diagonals* AC, BD *in* Y. *And* XeYE *be drawn. Then* 43.
EX :

Fig. 43. $EX : eX :: EY : eY$. *And if* E *be at an infinite diſtance, then* $eX = eY$.

For ſuppoſe BD, AC, to become diagonals (fig. 42.); and that EL, EK, EO, EM; and *el*, *ek*, *eo*, *em*, all lie in one line E*e*; and that AB, CD, interſect in X, and AC, BD, in Y; and that E*e* paſſes thro' XY; which is a particular caſe of this Prop. Then L, *l*, M, *m*, coincide in X; and K, *k*, O, *o*, in Y. Then the proportion, $EL \times EM : el \times em :: EK \times EO : ek \times eo$, becomes $EX^2 : eX^2 :: EY^2 : eY^2$; and $EX : eX :: EY : eY$. But if *e*E be infinite, then $EX = EY$, and $eX = eY$.

Cor. 2. *If* XA, XC, *be two tangents, and* AYC *paſſes thro' the points of contact. Then will* $EX : eX :: EY : eY$.

For let B approach to A, and D to C; then BD will coincide with AC. And it will ſtill be $EX : eX :: EY : eY$.

PROP. XXXIX.

44. *In any diameter, as its latus rectum, to the ſum of any two ordinates :: ſo their difference, to the difference of the abſciſſæ. Latus rectum of* AQ $: ND :: DC : MD$.

Let L = latus rectum; then (XVII.) $L \times AP = PM^2$, and $L \times AQ = QN^2$; and ſubtracting one from the other, $L \times AQ - L \times AP = QN^2 - PM^2 = \overline{QN + PM} \times \overline{QN - PM}$; that is, $L \times PQ = ND \times DC$. And $L : ND :: DC : PQ$.

Cor. *If* NC *be an ordinate to any diameter* AQ; MD *any other diameter. Then the rectangle* NDC $= MD \times$ *parameter of* AQ.

Fig.

PROP. XL.

If PM *be an ordinate to any diameter* AP, *and* 45.
AM *be drawn cutting any other ordinate* QO *in* N.
Then $PM \times QN = AQ \times$ *parameter of* AQ.

Let $L =$ latus rectum; then (XVII.) $QO^2 = L \times AQ$, and (XVI.) $PM^2 : QO^2$ or $L \times AQ :: AP : AQ ::$ (ſimilar triangles) $PM : QN :: PM^2 : PM \times QN$; therefore $L \times AQ = PM \times QN$.

Cor. 1. $PM \times QN = QO^2$.

Cor. 2. *Every line paſſing thro'* A, *except the diameter* AP, *will meet the curve again in ſome other point* M.

For $QN : QA :: L : PM$ the ordinate where it meets the curve again.

PROP. XLI.

If AQ *be any diameter, and any line* CF *be drawn* 46.
cutting it in C, *and the curve in* D, *and* F. *And*
if the ordinates DB, FQ *be drawn. Then* $CA^2 =$
$AB \times AQ$.

For (XVI.) $AB : AQ :: BD^2 : QF^2 ::$ (ſimilar triangles) CB^2 or $\overline{CA + AB}^2 : CQ^2$ or $\overline{CA + AQ}^2$; whence $AB \times \overline{CA^2 + 2CA \times AQ + AQ^2} = AQ \times \overline{CA^2 + 2CA \times AB + AB^2}$, and $AB \times \overline{CA^2 + AQ^2} = AQ \times \overline{CA^2 + AB^2}$. Therefore $\overline{AQ - AB} \times CA^2 = AQ^2 \times AB - AB^2 \times AQ = \overline{AQ - AB} \times AQ \times AB$. Therefore $CA^2 = AQ \times AB$.

Cor. *Any line that is not a diameter, drawn within the curve, will interſect it in two points.*

For

Fig. 46. For ſince DF is not a diameter, and therefore not parallel to AQ, it will interſect AQ ſomewhere as at C. Therefore by this Prop. it is AB : AC : : AC : AQ, the abſciſſa belonging to the ordinate paſſing thro' the other interſection F.

PROP. XLII.

47. *If* AB *be any diameter,* EG *an ordinate,* AB *a tangent at the vertex; and if* AG *be drawn. Then any line* BF *drawn parallel to the diameter* AE, *cutting the curve in* M, *and the line* AG *in* D; BM, BD, BF *will be continually proportional.*

For drawing the ordinate MP, and DC parallel to it. Then (XVI.) AP : AE : : PM^2 or CD^2 : EG^2 : : (ſimilar triangles) AC^2 : AE^2. Therefore AP : AC : : AC : AE; that is, BM : BD : BF.

PROP. XLIII.

48. *If* FG, HQ, *be ordinates to any diameter,* AQ; BFT *a tangent at* F; FL *a diameter paſſing thro'* F. *Then if* QH *be produced to the tangent* FB; *then* BH, HL, FM *are in continual proportion.*

Let p be the parameter of AQ; then (XX.) TG = 2AG, and by ſimilar triangles, 2AG : GF : : FL : LB : : $p \times LF$: : $p \times LB$. Then $p \times LF \times GF = p \times LB \times 2AG =$ (XVII.) $2FG^2 \times LB$, and $2FG \times LB = p \times LF =$ (XXXIX. Cor.) $HL \times LC$. Whence BL : LH : : LC : 2FG or FM; and by diviſion, BL — LH or BH : LH : : LC — FM or LH : FM.

Cor. 1. $BL \times FM = p \times FL = HLC$.

Cor. 2. *If* FL *be any diameter;* FB *a tangent at* F, *cutting the ordinate* QH *in* B. *Then* $BL^2 =$ *rectangle* HBC.

For draw the diameter AQ; then BH : HL : : HL : FM, and (by adding) BH : (BH + HL) BL : : HL : (HL + FM) HK. And (adding again) BH : BL : : (BH + HL) BL : (BL + HK) BC. Fig. 48.

PROP. XLIV.

If AC *be a tangent at* A, AH *any line drawn within the parabola,* DF *any diameter cutting the tangent in* B, *and the line* AH *in* F. *Then* AF : FH : : BD : DF. 49.

Draw the diameter GLO to the ordinate AG or GH, and draw HC parallel to DB, put p = latus rectum. Then (XVII.) $p \times GL = AG^2$, and $p \times 4GL = 4AG^2 = AH^2$. But $AG = \frac{1}{2}AH$; therefore $GO = \frac{1}{2}HC$ = (XX. Cor. 1.) 2GL, and $HC = 4GL$, and $p \times 4GL$ or $p \times HC = AH^2$. Therefore p : AH : : AH : HC : : (ſimilar triangles) AF : FB; and alternately AH : FB : : p : AF : : (XXXIX.) FH : FD : : AH — FH : FB — FD; that is, FH : FD : : AF : DB; or AF : FH : : DB : DF.

Cor. 1. *If* LG *be any diameter,* AH *an ordinate,* AC *a tangent at* A, HC *parallel to* GL; *then* $AH^2 = HC \times$ *parameter of* LG.

Cor. 2. *The ſame things ſuppoſed,* HC = 4GL.

Cor. 3. *The ſame things ſtill ſuppoſed,* $AF^2 = BD \times$ *parameter of* LG.

For AF : FH : : BD : DF. And compounding, AF : AH : : BD : BF. And by ſimilar triangles, AF : AH : : BF : CH. And multiplying, $AF^2 : AH^2$: : BD : CH. Whence $CH \times AF^2 = BD \times AH^2 = BD \times HC \times p$; therefore $AF^2 = p \times BD$.

Fig.

PROP. XLV.

50. *If* FT *be a tangent at* F, LI *parallel to it.* LB, FC, ID, *ordinates to any diameter* AD. *Then* LB + ID = 2FC, *when both* L *and* I *are on the ſame ſide of* AD.

Draw the diameter FK which will biſſect LI in G; and thro' L, draw the diameter LP. Then by ſimilar triangles, ſince IG = GL, IK = KP = FN. And by parallel lines, PD = NC = LB. Therefore adding equals to equals, IK + LB = FN + NC = FC. And doubling, IP + 2LB or ID + LB = 2FC.

Cor. *If* L *be on the other ſide of* AD; *then* ID — LB = 2FC.

For then ſubtracting equals from equals, it will be IK — LB = FN — NC = FC; and by doubling, IP — 2LB or ID — LB = 2FC.

PROP. XLVI.

51. *If* AF *be the axis,* F *the focus*; FD *perp. to* AF. *Draw any line* FM, *and* MD *parallel to* AF. *Then* FM — MD = FI, *half the parameter of the axis*; *when* D *is without the curve.*

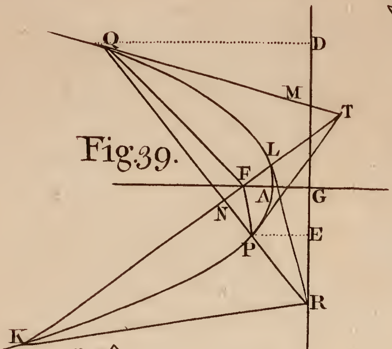
For make AC = AF, and draw the directrix CE. Then (Prop. I.) FM = ME = MD + DE = MD + CF = (Prop. II.) MD + FI.

Cor. *If* D *is within the curve,* F*m* + *md* = FI *half the parameter.*

For F*m* = *me* = *de* — *dm* = FI — *dm*. And F*m* + *dm* = FI.

PROP.

Fig.39.



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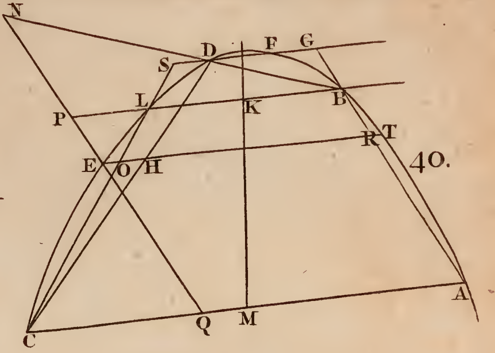

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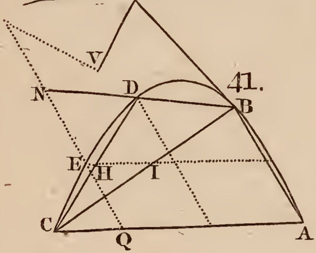

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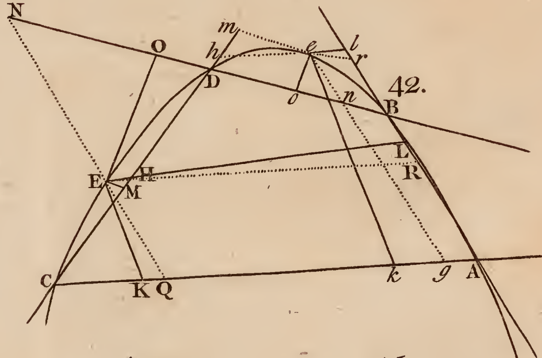

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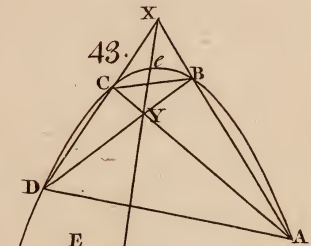

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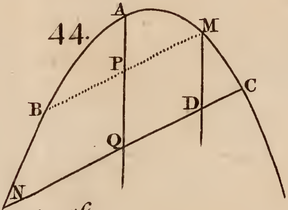

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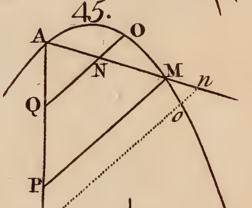

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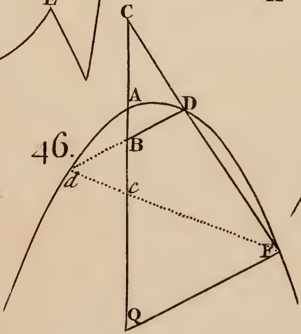

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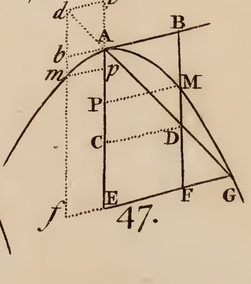

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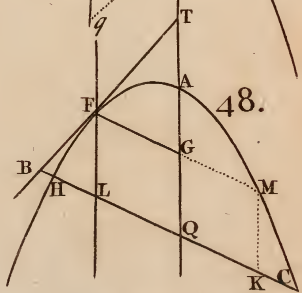


Pl.XXVIII. pa.196.

PROP. XLVII.

If any circle cut a parabola in four points B, C, D, E; *and from the points of interſection, ordinates be drawn to the axis* AI; *the ſum of the ordinates on one ſide of the axis, is equal to the ſum of the ordinates on the other*; CG + BI = EF + DH. 52.

Draw the lines BC, DE, joining the points by two and two, let them meet in N. Let PT, QS be the diameters of BC, DE; from the vertices of which let the tangents PR, QR, be drawn, which are parallel to BC, DE. Then (XXVI. Cor. 2.) $RP^2 : RQ^2 :: BNC : END$. But (Geom. IV. 21. Cor. 3.) BNC = END; therefore $RP^2 = RQ^2$, and RP = RQ. Therefore drawing PQ, and biſſecting it in O, RO will be perp. to PO or OQ, conſequently RO is the axis. But (XLV.) BI + CG = 2PO = 2QO = FE + HD.

And if C fall on the other ſide of AI, then (XLV. Cor.) BI — CG = 2PO = 2QO = HD + FE, and BI = CG + HD + FE.

Cor. 1. *If* E *and* D *coincide in* Q, *then* HD + FE = 2OQ.

For then HD + FE becomes OQ + OQ or 2OQ.

Cor. 2. *If* C, E *and* D, *coincide in* Q, *then* BI = 3OQ.

For then HD + FE + CG becomes OQ + OQ + OQ.

Cor. 3. *If* C *falls in* A *the principal vertex*; *then* BI = FE + HD; *or* BI = 2OQ, *if* E *and* D *coincide.*

PROP. XLVIII.

53. *If* CF, AG *be two diameters; and from any point* B *in* CF, *any line* BE *be drawn cutting the line* AC *in* D, *the diameter* AG *in* G, *and the curve in* L *and* E. *Then the rectangle* LDE = *rectangle* BDG.

For by ſimilar triangles, $BD : DG :: BC : AG ::$ (XXV. Cor. 3.) $LBE : LGE$; and $BD \times LGE = DG \times LBE$; that is, $BD \times LGE = DG \times LB \times \overline{BG + GE}$, or $BD \times GE \times \overline{LD + DG} = DG \times LB \times BG + DG \times LB \times GE$; and $BD \times GE \times LD + BD \times GE \times DG - DG \times LB \times GE = DG \times LB \times BG$. That is, $BD \times LD \times GE + LD \times DG \times GE = DG \times LB \times BG$, or $BG \times LD \times GE = DG \times LB \times BG$; therefore $LD \times GE = BL \times DG$; and adding $LD \times DG$, $LD \times DG + LD \times GE = LD \times DG + BL \times DG$; that is, $LD \times DE = BD \times DG$.

Cor. *The rectangle* $LD \times GE$ = *rectangle* $LB \times DG$.

PROP. XLIX.

54. *If* AB *be any diameter;* LBN *an ordinate;* LC, NC, *two tangents at* L *and* N; DG *any line parallel to the diameter* AB, *cutting the tangents in* D *and* O, *and the curve in* F. *Then* FG^2 = *rectangle* OFD.

For (Prop. XLIV.) $OF : FG :: LG : GN :: FG : FD$. Therefore $FG^2 = OF \times FD$.

PROP.

Fig.

PROP. L.

If PG, AO *be any two diameters*, PO *an ordinate to* AO. *And if any line* AF *be drawn from* A, *to cut* PO, PG *in* N *and* L, *and the curve in* F; *then* $AL^2 = AN \times AF$. 55.

For by reaſon of the parallels, LG, AD; AF : AL : : DF : DG : : (XXXIV.) DE : DF : : (Geom. II. 13. Cor.) OP : ON : : (by parallel lines) AL : AN; and $AF \times AN = AL^2$.

PROP. LI.

If LC, NC, DF, *be three tangents, at* L, N *and* A; *then will* DA : AF : : LD : DC : : CF : FN. 56.

Thro' D, A, C, and F, draw the diameters DG, AH, CI, FM. Draw LA, AN ordinates to DG, FM; ſince theſe ordinates are biſſected by their diameters, we ſhall have LG = GH, NM = MH, and alſo LI = IN; that is, LH + HI = NM + MI = MH + MI = 2MI + HI, and LH or 2GH = 2MI, and MI = GH = LG; and GI = HM = MN.

By reaſon of the parallels, DA : AF : : GH : HM : : or LG : GI : : (ſimilar triangles) LD : DC. For the ſame reaſon, DA : AF : : GH : HM : : or MI : MN : : CF : FN.

Cor. 1. *If the diameter* CRI *cut the tangent* DF *in* B; *then* BF = DA.

For GH = IM; therefore by reaſon of the parallels, DA = BF.

Cor. 2. *If* RI *be the axis, and* LN *an ordinate to it*; LC, NC, *tangents at* L *and* N, DAF *a tangent at any point* A; *then* LD = CF, *and* DC = FN.

For

Fig. 56. For in that caſe CI is perp. to LN, and $CL = CN$, and the angle $NCI = LCI$; therefore the right angled triangles LCI, NCI, LDG, NFM are ſimilar; whence $CF : IM :: FN : NM :: LD : LG$ or IM; therefore $CF = LD$, and conſequently $FN = CD$.

PROP. LII.

57. 58. *All parabolas are ſimilar figures.*

Two figures are ſimilar, when ſimilar right lined figures can always be inſcribed in them, being drawn in a like ſituation.

Take $BC : bc :: AL : al$; then $BC \times al = bc \times AL$, and $BC \times al \times AL = bc \times AL^2$. But (Prop. III.) $BC \times AL = DL^2$, and $bc \times al = dl^2$. Whence $DL^2 : dl^2 :: BC \times AL : bc \times al :: BC \times AL \times al$ or $bc \times AL^2 : bc \times al^2 :: AL^2 : al^2$. Whence $AL : al :: DL : dl$; therefore the triangles ALD, *ald* are ſimilar, and conſequently the curves DBA, *dba* are ſimilar.

Cor. *Hence two parabolas, that have the ſame parameter and abſciſſa, are both equal and ſimilar.*

PROP. LIII.

59. *If a parabola* CAD, *and ellipſis* AGB, *be deſcribed, having the ſame vertex and focus* A *and* F. *Then as the tranſverſe of the ellipſis : to the diſtance of the focus from the further vertex : : ſo the latus rectum of the parabola : to the latus rectum of the ellipſis.* $AB : BF :: CD : GH$.

For (Book I. Pr. VI. Cor. 2.) $BA : GH :: BF \times FA : GF^2$. But (Prop. II.) $AF = \frac{1}{4} CD$; and $GF^2 = \frac{1}{4} GH^2$. Therefore $BA : GH :: BF \times \frac{1}{4} CD :$

$\frac{1}{4}$ CD : $\frac{1}{4}$ GH^2 : : BF × CD : GH^2; therefore BA × GH^2 = BF × CD × GH; and BA × GH = BF × CD. Fig. 59.

Cor. *If a parabola and ellipſis has the ſame focus and latus rectum. The diſtances of the vertices from the focus will be in the ſame ratio.* AB : BF : . AF : FO.

For BA : GH : : BF × FA : $\frac{1}{4}$GH^2. But GH = 4OF; therefore BF × FA : 4OF^2 : : BA : 4OF : : BA × OF : 4OF^2; therefore BF × FA = BA × OF.

PROP. LIV.

Two equal parabolas upon the ſame axis, but having different vertices, approach infinitely near one another, but never meet. 60.

Let p be the parameter of the two parabolas DAL, FBC. Then (Prop. III.) $p \times \overline{AB + BG} = DG^2$, and $p \times BG = FG^2$; and ſubtracting, $p \times AB = DG^2 - FG^2$ = rectangle DFC. Now ſince $p \times AB$ is a given quantity, and ſince the ordinate FG continually increaſes with the axis BG, FG and FC at laſt become infinitely great, and therefore DF is infinitely little.

Cor. *In the parabola the ordinate continually increaſes, as the abſciſſa increaſes; but in a leſs ratio; and when the curve is infinitely extended, the abſciſſa becomes infinitely greater than the ordinate.*

For $p \times AG = DG^2$, whence p : DG : : DG : AG, ſo that the greater AG or DG is, the leſs is the ratio of p to DG or of DG to AG; and when AG becomes infinite, then $p \times AG$ is infinite; therefore DG^2 is infinite, and DG is infinite; and is therefore infinitely greater than p; and conſequently infinitely leſs than AG.

Fig.

PROP. LV.

61. *The area of a parabola* FAG *is* $\frac{2}{3}$ *the area of the circumſcribing parallelogram, or* $\frac{2}{3}$ AB × FG.

Draw AD, DG parallel to BG, AB; divide AD into an infinite number of equal parts, and thro' all the points C, ſuppoſe lines, as CN, to be drawn parallel to AB. Then according to the method of indiviſibles, all the lines CN, compoſe the area ANGD. Put AP $= x$, PN $= y$, $r =$ parameter; then by the nature of the curve (Prop. XVII.) $rx = yy$, and $x = \frac{yy}{r} =$ CN, and the ſum of all the CN or of all the $\frac{yy}{r}$ (by Prop. III. Arithmetic of infinites) is $\frac{1}{3} \times \frac{y^3}{r} =$ area ANC; and putting x for $\frac{yy}{r}$, the area ANC $= \frac{1}{3} xy = \frac{1}{3}$ AC × CN, and conſequently the area ADG $= \frac{1}{3}$ AD × DG. Whence the area ANGB $= \frac{2}{3}$ AD × DG or $\frac{2}{3}$ AB × BG. And the area FAG $= \frac{2}{3}$ AB × FG.

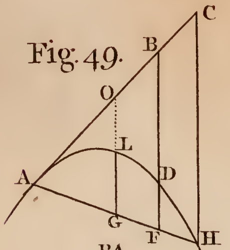
Otherwiſe,

Suppoſe AB divided into an infinite number of equal parts, thro' which draw the ordinates as PM. Then ſince $rx = yy$, we have y or PM $= \sqrt{rx} = \sqrt{r} \times x^{\frac{1}{2}}$. And (Prop. V. Arith. Infinites) the ſum of all the PM's is $= \frac{\sqrt{r} \times x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}\sqrt{rx} \times x = \frac{2}{3} yx$ for the area APM. Whence the area ABF $= \frac{2}{3}$ AB × BF, and the area FAG $= \frac{2}{3}$ AB × FG.

Cor. *The area of the complement* ANGD $= \frac{1}{3}$ AB × BG.

PROP.

Fig. 49.



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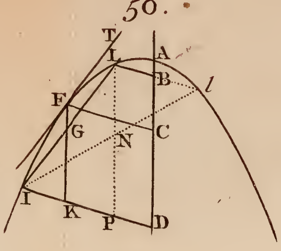

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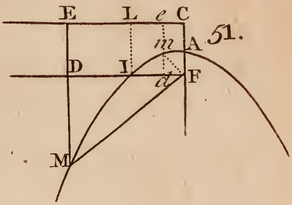

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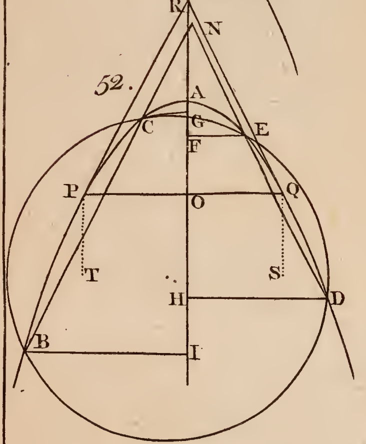

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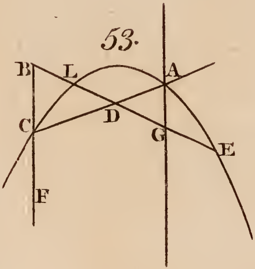

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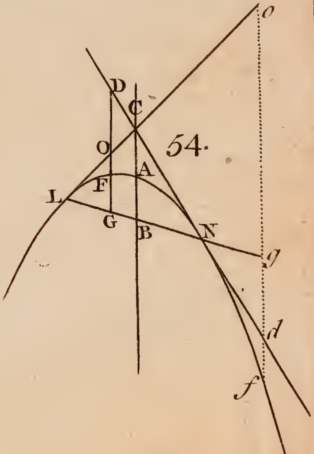

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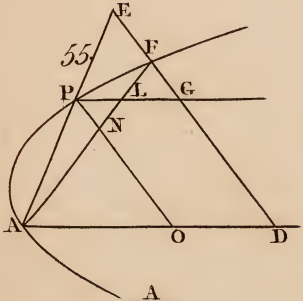

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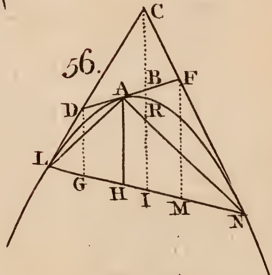

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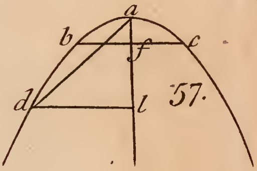

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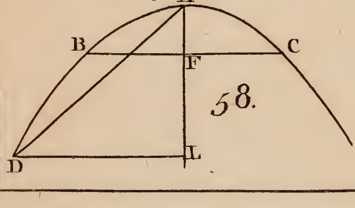

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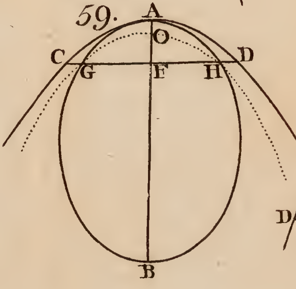

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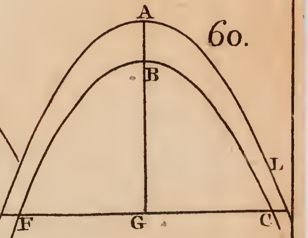


Pl. XXIX. pa. 202.

PROP. LVI.

A parabolic conoid is $\frac{1}{2}$ *its circumſcribing cylinder.* 61.

Let AB be the axis, and ſuppoſe it divided into an infinite number of equal parts; thro' which let circular planes be drawn as MN, then all theſe circles make up the ſolid. Let AP $= x$, PM $= y$, parameter $= r$, $c = 3.1416$. Then (XVII.) $rx = yy$, and $crx = cyy =$ circle MN. And (Prop. II. Arith. of Infinites) the ſum of all the $crx = \frac{crx}{2} \times x = \frac{1}{2} crxx = \frac{1}{2} cxy^2 = \frac{1}{2}$ AB × circle FG.

Cor. *The ſolidity of the fruſtum* FMNG *is equal to a cylinder whoſe hight is* PB *the hight of the fruſtum; and baſe, half the ſum of the baſes of the fruſtum* $= \frac{\textit{circle}\ \mathrm{FG} + \textit{circle}\ \mathrm{MN}}{2} \times \mathrm{PB}$.

Let F = fruſtum, PB $= h$, FB $= b$, then F $= \frac{cbb}{2} \times \overline{x + h} - \frac{cyy}{2} x$. But (XVI.) $x : yy :: x + h : bb$, and $bbx = yyx + yyh$, therefore $2\mathrm{F} = (cbbx + cbbh - cyyx =)\ cyyx + cyyh + cbbh - cyyx = cyyh + cbbh$, and $\mathrm{F} = \frac{cbb + cyy}{2} \times h$.

PROP. LVII.

The content of a parabolic ſpindle, or of the ſolid made by the parabola revolving round the ordinate BG, *is* $\frac{8}{15}$ *the circumſcribing cylinder.* 62.

Let BG $= a$, AB $= b$, GP $= x$, PM $= y$, r = parameter, $c = 3.1416$. Then (IV. Cor. 1.) PM $\times r = \overline{\mathrm{BG} + \mathrm{BP}} \times$ PG, or $ry = 2ax - xx$, and $y =$

Fig. 62. $y = \frac{2ax - xx}{r}$, and $cyy = \frac{4aaxx - 4ax^3 + x^4}{rr} c =$ circle PM.

Then if GB be divided into an infinite number of equal parts, and circles as PM drawn through them, the ſum of all theſe conſtitute the ſolid. But (Arith. Inf. Prop. III.) the ſum of all the $\frac{4aa}{rr} xx = \frac{4aa}{3rr}x^3$, and (ib. IV.) the ſum of all the $\frac{4a}{rr}x^3 = \frac{a}{rr}x^4$, and (V.) the ſum of all $\frac{x^4}{rr} = \frac{x^5}{5rr}$. Therefore the content of the ſolid GPM $= \frac{c}{rr} \times : \frac{4}{3}aax^3 - ax^4 + \frac{1}{5}x^5$. And when x becomes BG or a; then the whole ſolid $= \frac{c}{rr} \times \frac{4}{3}a^5 - a^5 + \frac{1}{5}a^5 = \frac{c}{rr} \times \frac{8}{15}a^5$; but $rb = aa$, and $rrbb = a^4$. Therefore, ſolid ABG $= \frac{c}{rr} \times \frac{8}{15}rrbba = \frac{8}{15}cbba = \frac{8}{15}$ circle AB × GB.

61. Cor. *The ſegment of a parabolic ſpindle* ANOB *is equal to a cylinder, whoſe baſe is eight times the greater baſe* AB, *together with three times the leſſer baſe* NO, *and alſo four circles which are mean proportionals between the two baſes; and the hight* $\frac{1}{15}$ *the hight of the fruſtum.*

For let AP $= x$, PN $= y$, AB $= b$, $c = 3.1416$, NO $= t$. And $rx = yy$; then $x = \frac{yy}{r} =$ CN, and the area deſcribed by CN revolving about BO is $= c \times \mathrm{OC}^2 - c \times \mathrm{ON}^2 = cbb - c \times \overline{b - x}^2 = 2cbx - cxx = \frac{2cbyy}{r} - \frac{cy^4}{rr}$. Therefore if

AC

Fig. 61. AC be divided into an infinite number of equal parts, and planes drawn thro' them, the ſum of all the circular annuli compoſing the ſolid ACN $= \frac{2cby^3}{3r} - \frac{cy^5}{5rr}$. This taken from the cylinder ACOB or $cbby$, leaves $cbby - \frac{2cby^3}{3r} + \frac{cy^5}{5rr} = cbby - \frac{2}{3}cbyx + \frac{1}{5}cxxy = cbby - \frac{2}{3}cby \times \overline{b-t} + \frac{1}{5}cy \times \overline{b-t}^2 = \frac{8}{15}cbby + \frac{4}{15}cbyt + \frac{3}{15}cytt.$

PROP. LVIII. *Prob.*

To deſcribe a parabola by continued motion. 63.

1 *Way.*

Having a diameter, the latus rectum, and the angle the tangent makes with the diameter. Produce the diameter PA, till AG be equal to the latus rectum; draw the tangent AL, and make the angle AGD = GAL = BAC. Then if the angle BAC be moved about the fixed point A, whilſt the leg AB carries with it the end D of the ruler DM, along the line GD, and DM keeps always parallel to GA. Then the interſection of DM with the leg AC, at M, will deſcribe the parabola AMF.

Draw the ordinate MP parallel to AL, the lines PM, GD are equal, as being equally inclined to the parallels GP, DM. The triangles AGD, APM are ſimilar; for MPA = GAL = AGD, and ſince GAL = BAC or DAM, take away DAL, then GAD = LAM = AMP. Whence AG : GD : : PM : AP; that is, latus rectum : PM : : PM : AP, and AP × latus rectum $= \text{PM}^2$; therefore (XVII.) AMF is a parabola.

If AP is the axis, then the angles GAL, AGD, DAM, APM, are right angles. Then it appears

Fig. 63. at once, that DL or AG : AL or PM : : AL or PM : LM or AP.

2 *Way*.

64. Let AP be a diameter, AC a tangent at A. Produce the diameter AP, and make the iſoceles triangle HBL, where BL is parallel to AC, and draw HLIQ; draw HF parallel to AC, and equal to the latus rectum; alſo draw ID parallel to AP. Then if the figure FHBLQ be ſo moved that HB always ſlides along the diameter HA, and HQ moves the line ID along CA, always parallel to itſelf, whilſt the point F turns the line GK about the fixed point A. Then the interſection M, of the lines AK, ID, will deſcribe the parabola AOM.

For draw the ordinate PM. Then ſince HB = BL; therefore HA = AI = PM. And the triangles AHF, APM being ſimilar, it will be AH or PM : HF : : AP : PM. Whence HF × AP = PM^2; therefore (XVII.) the point M is in the parabola.

SCHOLIUM.

A parabola may likewiſe be deſcribed by Def. 1.

PROP. LIX. *Prob.*

The axis and parameter being given, to deſcribe a parabola; by finding ſeveral points in it.

1 *Way*.

65. Let AC be the axis, make AF = AB = $\frac{1}{4}$ the latus rectum, and draw the directrix BD perp. to BC, and F is the focus.

From any point G (above B,) as a center, with the diſtance GF, croſs DH at D and H; with the ſame diſtance, and from the centers D and F, deſcribe two arches cutting one another in K. Likewiſe from H and F, deſcribe two arches cutting each other in L. Then K, L are two points of the curve. And ſo for more points. Fig. 65.

For FGDK is an equilateral parallelogram, and KD = KF. Therefore (Prop. I.) K is in the curve; and L likewiſe.

2 *Way.*

Let AC be the axis, make AF = AB = $\frac{1}{4}$ latus rectum. Draw KCL perp. to AC. From the center F with the radius BC, cut KL, in K and L, which are two points of the curve. And the like for more points. 65.

For ſince BC = FK = FL; therefore (I. Cor. 2.) K and L are in the curve.

SCHOLIUM.

When the parameter is not given, draw an ordinate; and find a third proportional to the abſciſſa and ordinate, which will be the parameter. The reſt as before.

3 *Way.*

Let AE be the axis, F the focus; let AC = CS = latus rectum, and CS perp. to AE, draw alſo AD perp. to AE. Divide CS into any number of equal parts, and AC into as many equal parts as is the ſquare of CS. Then from the ſcale CS, ſet along AD, 1, 2, 3, 4, &c. parts; thro' all which points draw parallels to AE; and on theſe parallels, from the ſcale AC, ſet off 1, 4, 9, 16, &c. parts, viz. all the ſquares of the former numbers, having their origin at AD. And thro' the points where they terminate, draw a regular curve for the parabola, as A1G3S5. 66.

For let CS = n parts, and AC = nn parts. Therefore the parts in AC to the parts in CS, will be as n to 1. Now (Prop. II.).

FG = $\frac{1}{2}$ CS = $\frac{1}{2}n$ parts of the ſcale CS;
and AF = $\frac{1}{4}$ AC = $\frac{1}{4}nn$ parts of the ſcale AC = $\frac{1}{4}n$ parts of the ſcale CS.

There-

Fig. Therefore $FG : FA :: \frac{1}{2}n : \frac{1}{4}n :: 2 : 1$, as it
66. ought by Prop. II. And ſince the ordinates 11, 2G, 33, &c. are as the ſquares of the abſciſſa A1, A2, A3, &c. therefore (III. Cor. 2.) all the points 1, G, 3, S, &c. are in the curve of the parabola.

PROP. LX. *Prob.*

67. *Given any diameter, and the tangent at the vertex; to deſcribe the parabola, by finding a number of points in it.*

1 *Way.*

Let AB be the diameter, AL a tangent; and if the parameter be given, produce PA to G, ſo that AG may be equal to the parameter; biſſect GA in C. Draw AF perp. to AP. Take any point O, and with radius OG, and center O, cut the lines AP, AF in P and F. Thro' P draw MM parallel to AL, and make PM, PM = AF. Then M, M, will be two points in the curve.

For GP is the diameter of a circle whoſe ordinate is AF, and therefore $GA \times AP = AF^2$; that is by conſtruction, $AP \times$ latus rectum $= PM^2$; therefore (XVII.) M is in the parabola. And thus as many points as you will may be found, and then the curve drawn thro' theſe points.

2 *Way.*

68. If the diameter AP, tangent AL, a point M in the curve, be given. Thro' M draw ML parallel to AP, cutting the tangent AL in L. Prolong PA upwards, and make AF = LB. Draw AB and MF, their interſection N, will be in the curve of the parabola. And thus any number of points may be found.

This conſtruction is evident from Prop. XXIV.

Fig. 68.

SCHOLIUM.

The parameter is found, by ſeeking a third proportional to the abſciſſa and ordinate when drawn.

3 *Way*.

Let AP be the diameter, GAL the tangent. 69.
Make AG = to the parameter, and draw FG parallel to AP. Make GF = AL, and draw LM parallel to AP. Thro' A draw FM to interſect LM in M, a point in the parabola; and ſo for more points.

For draw FH parallel to GA, and the ordinate PM, and draw HL. Then LA = FG = HA. And the triangles APM, AHF are ſimilar, and AP : PM : : AH : HF; that is, by conſtruction AP : PM : : AL or PM : AG the latus rectum.

Therefore (XVII. Cor. 1.) M is in the curve of the parabola.

4 *Way*.

Let AB be the diameter, AL a tangent. Pro- 70.
duce BA, and make AE equal to $\frac{1}{4}$ the latus rectum. Thro' E draw ED perp. to EB, cutting the tangent AL in L. Make the angle LAF = LAE, and AF = AE. Thro' F draw DHC parallel to EB; then to the axis DC, and focus F, and directrix DE, deſcribe a parabola, by Prop. LIX.

For biſſect DF in H, then (I. Cor. 1.) H is the vertex, HC the axis; and ſince AF = AE, and angle FAL = LAE; therefore (Prop. I. and VIII. Cor. 2.) AH is a parabola.

SCHOLIUM.

If the abſciſſa and an ordinate are given. The parameter is found, by ſeeking a third proportional to the abſciſſa and ordinate.

The reſt as before.

PROP.

Fig.

PROP. LXI. *Prob.*

71. *A right line* DE *being given in a parabola; to find its diameter, the axis, parameter, and focus.*

Draw HI parallel to the given line DE. Biſſect DE, HI, in O and G, thro' which draw AOG for the diameter.

Draw HF perp. to AG and biſſect it in B; and draw MB parallel to AG, for the axis.

Make MB : HB : : HB : parameter of the axis.

Then $\frac{1}{4}$ the parameter let from M to S gives the focus S.

For ſince DE, HI are biſſected by AG; therefore (XV. Cor. 5.) DE, HI are ordinates to the diameter AG. And ſince MB is a diameter (being parallel to AG), and biſſects HF at right angles; therefore (IX. Cor. 2.) MB is the axis, and HB, BF, ordinates. The reſt is plain from Prop. XVII. Cor. 1. and Prop. II.

SCHOLIUM.

The parameter of the diameter AG is found in the ſame manner, by ſaying as AG : GH : : GH : parameter ſought.

PROP. LXII. *Prob.*

72. *Any diameter being given in a parabola; to draw an ordinate from a given point in the curve.*

1 *Way.*

Let AB be the diameter, H the point given; thro' H draw HC parallel to AB; and at the ſame diſtance from AB, on the other ſide, draw DL parallel to AB, to cut the curve in L; draw HL, which will be a double ordinate to AB.

For the parallels being equidiſtant, makes HF = FL. Fig. 72.

2 *Way.*

From H draw any line HK cutting AB in G; make GK = GH; thro' K draw KL parallel to AB, cutting the curve in L. Draw HL, which will be a double ordinate to the diameter AB.

For HG = GK; therefore by reaſon of the parallels, HF is = FL.

PROP. LXIII. *Prob.*

To find a diameter in a parabola, that makes a given angle with its ordinates. 73.

Draw any diameter DO (LXI), and make the angle ODC equal to the angle given, ſo that DC cut the curve in C. Biſſect DC in B, and draw the diameter AB parallel to DO, for that required.

For DB and BC are ordinates to the diameter AB, and the angle DBA = ODB = the angle given.

Cor. *There will be two diameters which anſwer this Prob. and equally diſtant from the axis; except the given angle be right, and then the axis is the diameter ſought.*

PROP. LXIV. *Prob.*

To any given point M *of a parabola, to draw a tangent.* 74.

1 *Way.*

Find any diameter AB (LXI.), and thro' M draw MC parallel to AB. Find an ordinate DC to the diameter MC (LXII.); thro' M draw MT parallel to DC, ſo is MT the tangent ſought.

This appears by Def. VII.

2 *Way.*

75. Draw any diameter AB, from M draw an ordinate to it MB (LXII.); make AT = AB, and draw TM the tangent at M.

This is plain from Prop. XX.

3 *Way.*

76. Draw MF from the given point to the focus; and draw ME parallel to the axis AB. Make ME = MF, and draw FE. Then draw MT parallel to FE, for the tangent in M.

For in the iſoceles triangle EMF, $< E = < F$; and ſince MD is parallel to EF, $<$ DMF = MFE = MEF = TME. Therefore (VIII. Cor. 3.) MT is a tangent at M.

4 *Way.*

From the given point M, draw MG parallel to the axis AB; and make MG = MF. Draw GF and biſſect it in D; draw MD for the tangent at M.

By Prop. VIII. and Cors.

5 *Way.*

77. Having the directrix CG and the focus F; draw MF from the given point to the focus; draw FT perp. to MF, to cut the directrix in T. Then draw MT, which will touch the curve in M.

By Prop. XXXIII. Cor. 3.

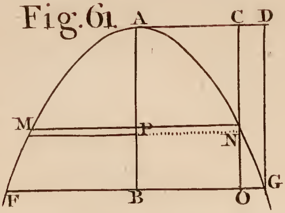
PROP. LXV. *Prob.*

78. *From a point given without the parabola, to draw a tangent to it.*

1 *Way.*

Thro' the given point T, draw the diameter AB; and make AB = AT. From any point D in the curve,

Fig.61.



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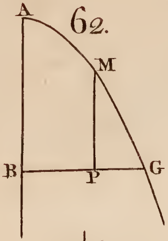

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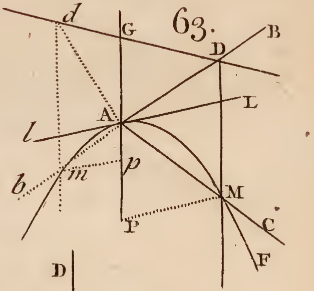

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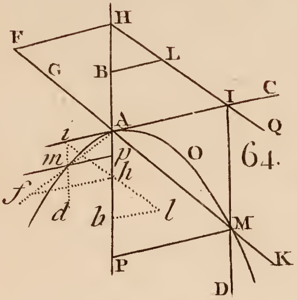

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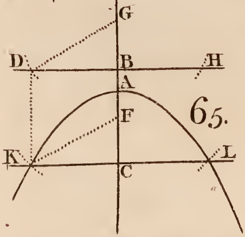

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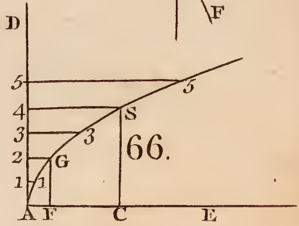

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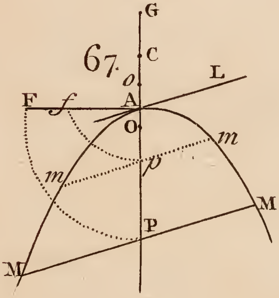

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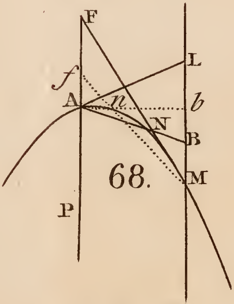

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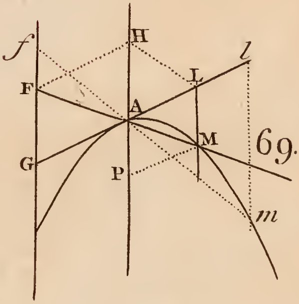

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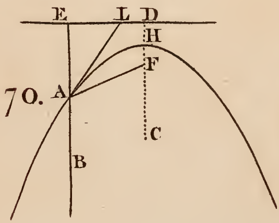

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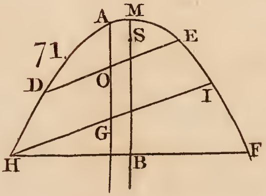

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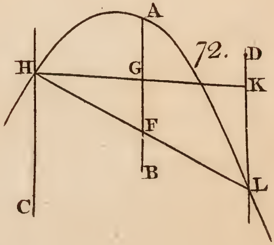


Pl.XXX.*pa.212.*

curve, draw the ordinate DC (LXII.); and thro' B Fig. draw MN parallel to DC. Then MT, NT, drawn, 78. will be tangents at M and N.

This appears by Prop. XX.

2 *Way.*

From the given point T draw any line cutting 79. the curve in H and C. Find FL a mean proportional between TH and TC. Thro' L draw the diameter FL cutting the parabola in F, the point of contact.

This is plain by Prop. XLIII. Cor. 2.

PROP. LXVI. *Prob.*

Given the focus F *of a parabola, and three points* 80. *in the curve* M, N, P; *to deſcribe the parabola.*

1 *Way.*

Thro' N, M, and alſo N, P, draw the lines NK, NI. And draw MF, NF, PF, to the focus. Then make FN : FM : : NK : MK; and FN : FP : : NI : PI. Thro' K and I, draw the line KI, and FT perp. to it from the focus. Biſſect FT in A, for the vertex. Then with the focus F, and directrix IK, deſcribe a parabola thro' the point M (Prop. LIX.), which will paſs thro' the points N, P.

Draw NG, PL, MH perp. to KT. Then by conſtruction, FN : FM : : NK : MK : : (ſimilar triangles) NG : MH. Alſo FN : FP : : NI : PI : : (ſimilar triangles) NG : PL. Therefore MF, NF, PF are to one another, as MH, NG, PL. Conſequently, ſince MF = MH; therefore NF = NG, and PF = PL. And all the points M, A, P, N, (Prop. I.) are in the parabola.

2 *Way.*

From N thro' M and P, draw the lines NO, NB; 81. thro' F draw NH, alſo draw MF, PF. Draw FO

Fig. biſſecting the angle MFH, cutting NM in O. Alſo
81. draw FB biſſecting the angle PFH, to cut NP in
B. Thro' O and B draw the line OB. Then with the focus F and directrix OB, deſcribe a parabola thro' the point M, which will alſo paſs thro' the points N, P.

For (Prop. XXXV. Cor.) if N, M, be two points in the parabola; FO will cut NM in the point O of the directrix OB. And if N, P, be two points in the parabola; FB will cut NP in the point B of the directrix OB. Therefore the directrix and focus remaining the ſame, all the three points M, N, P, are in the ſame parabola.

SCHOLIUM.

After a like manner an ellipſis or hyperbola whoſe focus is given, may be deſcribed thro' three given points; by the help of Prop. LXIII. B. I. and Prop. LIX. B. II.

PROP. LXVII. *Prob.*

82. *To deſcribe a parabola thro' three given points, and whoſe diameters ſhall make a given angle with the ordinates.*

Thro' two of the points M, N, draw the right line MN, and thro' the point P, draw the diameter PF, making the angle PFM = angle given. Thro' G the middle of MN, draw the diameter AG parallel to PF. Then make, as rectangle MFN : MG^2 : : PF : AG. Then make as AG : GM : : GM : AH the parameter of AG. Laſtly, deſcribe a parabola with the diameter AG, and latus rectum AH, this will paſs thro' the points M, P, N, as required.

For ſince MFN : MGN : : PF : AG; therefore (XXV. Cor. 3.) the points M, P, N, are in the parabola. And $\angle$ AGM = PFM = angle given. Fig. 82.

PROP. LXVIII. *Prob.*

To draw a tangent to a parabola parallel to a given line CD. 83.

Draw the directrix AB; thro' the focus F, draw BH perp. to the given line CD, to cut the directrix in B. Draw BP parallel to the axis AF. Biſſect BF in G, and draw GP parallel to CD, cutting BP in P, and GP will touch the parabola in P.

For draw PF; then the triangles BPG and FPG are ſimilar and equal. For the angles BGP and FGP are right angles, being equal GHD. And BG = GF, and PG common, therefore PB = PF, whence (Prop. I.) P is in the parabola, and (VIII.) PG is a tangent.

PROP. LXIX. *Prob.*

About a given line as a diameter, to deſcribe a conic ſection, to paſs thro' a given point P; *and ſo as the ordinates may make a given angle with the diameter.* 84.

Let AB be the diameter, from P draw PQ making the given angle with AB. Biſſect AB in C, and draw CD parallel to PQ. Then if Q fall between A and B, make as rectangle AQB : QP^2 : : $AC^2 : CD^2$. And with the ſemidiameters AC, CD, deſcribe the ellipſis ADPB.

But if Q fall below B, with the ſemidiameters AC, CD, deſcribe the hyperbola BPF.

Laſtly, if AB do not terminate, but is infinitely extended towards B; make as AQ : QP : : QP : L, the latus rectum of AQ. Then with the diameter AQ, and latus rectum L, deſcribe the parabola AP, for that required.

Fig. 84. All this proceſs is plain from the properties of theſe figures before laid down.

PROP. LXX. *Prob.*

85. 86. *To deſcribe a conic ſection to paſs thro' five given points*, A, B, C, D, E.

Draw lines between the points A, C; and between the points B, D; to interſect in R. And thro' the fifth point E, draw EK, EH parallel to BD, AC; meeting them in G and Q. Then in the lines EK, EH, find the points K, H, ſo that rectangle AGC : KGE : : ARC : BRD : : EQH : BQD; taking QH, GK, towards ſuch parts as the problem requires.

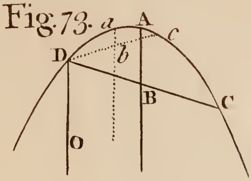
Biſſect KE, BD in L, and N, and draw LN. Alſo biſſect AC, EH, in P and O. And draw OP to interſect LN in F the center. But if LN, OP are parallel, the curve will be a parabola.

Produce FP, FO (if neceſſary) to I and M, ſo that IF = FM, and that $PA^2 : PA^2 - OE^2 :: FI^2 - FP^2 : OF^2 - FP^2$, to this add FP^2, and we have IF^2 and conſequently IF or FM, and IM the diameter to the ordinates AP, EO. And the ſection may be drawn by Prop. LXIX.

For ſince $IF^2 - FP^2 : FO^2 - FP^2 :: PA^2 : PA^2 - OE^2$, by compounding $IF^2 - FP^2 : IF^2 - FO^2 :: PA^2 : OE^2$;
that is, $IPM : IOM :: PA^2 : OE^2$; (therefore by the properties of the conic ſections) IM is the diameter of the ordinates PA, OE. The reſt appears by Prop. XLIII. Cor. 2. B. I.

The ſame demonſtration ſerves for the hyperbola, mutatis mutandis; and by Prop. XXXIX. Cor. 2. B. II.

Fig. 73.



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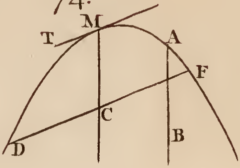

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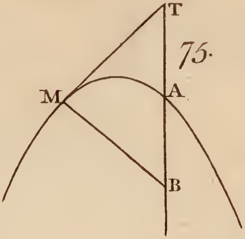

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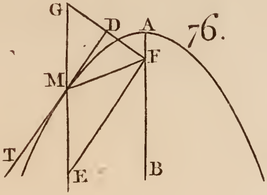

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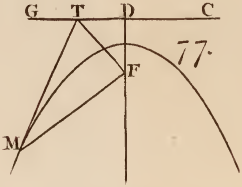

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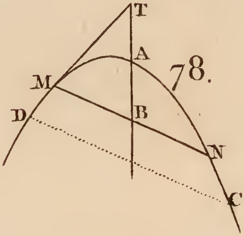

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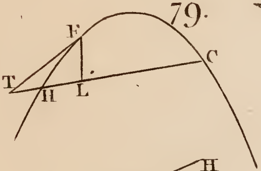

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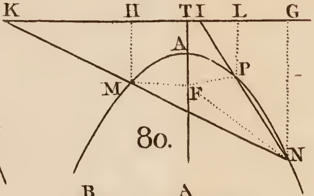

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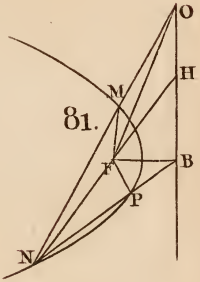

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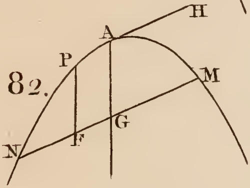

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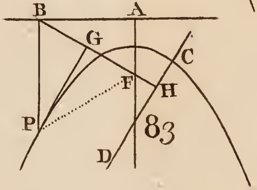

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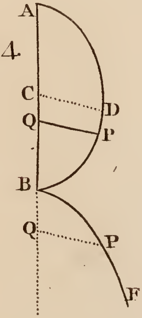

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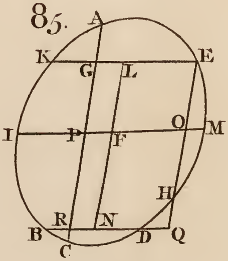

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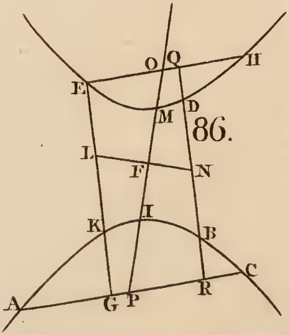


Pl.XXXI. *pa.216.*

Fig.

Or thus by points.

Draw AB, BC, DA, CD, forming the inſcribed trapezium ABCD, (or if A or C be infinitely diſtant, draw them in given directions). From E draw EF, EG, parallel to AB, AD. In EF, EG, take H and N, either both the ſame way, or both the contrary ways, from F and G, in reſpect of E; and ſo that EF : EG : : EH : EN. And draw BH, DN, to interſect in M a point of the curve. And thus innumerable points may be found, and the curve drawn thro' them. 87.

If BH, DN be parallel, then M is infinitely diſtant; or they may be parallel to the aſſymptotes of a hyperbola, or to the diameters of a parabola.

The practice is expeditiouſly performed thus; in EF and EG, take as many equal parts as you will, at 1, 2, 3, 4, &c. ſo that the magnitude of theſe in EF, to thoſe in EG, may be as EF to EG. Then thro' all the points in EF, draw lines from B; and thro' all the points in EN, draw lines from D, and where the correſpondent lines interſect, will be ſo many points of the curve.

For (Prop. LXX. Cor. 5. B. I.) ſince EF : EG : : EH : EN, then BH, DN, drawn will meet at M in the curve. And the like for the hyperbola (Prop. LXIII. Cor. 5. B. II.). And for the parabola (XXXVII. Cor. 5. B. III.)

Cor. 1. *If fewer points be given, provided others may be found to make up five; the curve will be deſcribed in the ſame manner.*

Cor. 2. *If four points be given, and the curve is to paſs thro' theſe points, and touch a right line given in poſition, in one of theſe points. It is only ſuppoſing two points to coincide in one; and then the line drawn thro' them, becomes a tangent to that point. And then the conſtruction will be the ſame as before.*

Cor.

Fig. 87. Cor. 3. *Likewiſe if the curve is to paſs thro' three points, and touch two right lines, given in poſition, in two of theſe points. The conſtruction will be the ſame, by ſuppoſing, two and two of theſe points to coincide.*

PROP. LXXI. *Prob.*

88. *To draw a conic ſection thro' four given points* B, C, D, E, *to touch a right line* HI *given by poſition.*

Draw lines thro' two and two as CE, DB, meeting in G, and cutting the tangent in I and H. Take GK a mean proportional between GE and GC; IP a mean between IC and IE; GN a mean between GB and GD; and HL a mean between HD and HB, and HQ a mean between HO and HF. Then in the tangent HI, find the point A, ſo that HA : AI : : GK × HL : IP × GN; and A will be the point of contact. Therefore by Prop. LXX, thro' the five points A, B, C, D, E, deſcribe the conic ſection, and it is done.

For (XLIII. Cor. 1. B. I. &c.) rectangle BGD : EGC : : DHB : OHF; that is, $GN^2 : GK^2 :: HL^2 : HQ^2$, and GN : GK : : HL : HQ; whence GN × HQ = GK × HL. Again XLIV. Cor. 3. B. I. &c.) $AH^2 : AI^2 :: OHF : CIE :: HQ^2 : IP^2$; and AH : AI : : HQ : IP : : HQ × GN : IP × GN : : GK × HL : IP × GN.

According to the various ſituation of the points C, D, E, B, and tangent HI; an ellipſis, hyperbola or parabola will be deſcribed.

PROP.

PROP. LXXII. *Prob.*

To deſcribe a conic ſection thro' three given points, 89.
B, C, D; *to touch two right lines given by poſition,* HI, NK.

Thro' two of the points B, D, draw BD meeting the tangents in H and K. And thro' two others of the points, C, D, draw CD, alſo meeting the tangents in I and L. Take four lines T, V, X, Y, ſo that TT = rectangle BHD, VV = BKD, XX = CID, YY = CLD. Then in the lines HD and ID, find the points R, S, ſo that HR : KR : : T : V, and IS : LS : : X : Y. Draw RS cutting the tangents in A and E, the points of contact. Then thro' the five points A, B, C, D, E, draw a conic ſection by Prop. LXX.

Draw IF parallel to KN, meeting the curve in O and F. And make IZ^2 = OIF. Then ſuppoſing A, E, the points of contact, (XLIV. Cor. 3.) rectangle OIF or IZ^2 : LE^2 : : CID or XX : CLD or YY. And IZ : LE : : X : Y : : (conſtruction) IS : LS. Therefore the points S, E, Z, are in one ſtreight line. Again let the tangents meet in G, then rectangle OIF or IZ^2 : IA^2 : : GE^2 : GA^2. And IZ : IA : : GE : GA; therefore the points E, Z, A are in one right line. And by the ſame reaſoning the points E, R, A, are in one right line. Therefore the points A, E, are in the right line RS. That is, the points of contact lie in the right line RS, as determined in the conſtruction.

Fig.

PROP. LXXIII. *Prob.*

90. *To draw a conic ſection thro' two given points* D, E; *to touch three right lines*, TG, TH, CH, *given in poſition.*

Thro' the given points D, E, draw the line FQ, to cut the tangents in G, Q, and P. And let XX = EQD, YY = DGE, and ZZ = EPD. And make, as FQ : FG : : X : Y. And GK : PK : : Y : Z. And from F draw FH to the interſection of the tangents TH, CH. And make FI : FH : : IM : MH. Thro' K, M, draw a line interſecting the tangents AT, CH, in A and C the points of contact. Thro' F and A, draw a line interſecting the tangent TH in B the point of contact. Then thro' A, B, C, D, E, draw a conic ſection, by Prop. LXX.

For ſince GK : PK : : Y : Z, therefore GK^2 : PK^2 : : Y^2 or DGE : ZZ or EPD; therefore (Prop. LIV. B. I.) K is in the line joining the points of contact A and C. Alſo ſince FQ : FG : : X : Y, and FQ^2 : FG^2 : : EQD : DGE; therefore (LIV. B. I.) as FQ cuts the tangents TG, TQ, in G and Q, F will be in the line AB, joining the points of contact A and B. Alſo for the tangents TG, HC; IM^2 : MH^2 : : LIN : LHN. And for the tangents TG, TH; FI^2 : FH^2 : : LIN : LHN. Therefore IM^2 : MH^2 : : FI^2 : FH^2. And IM : MH : : FI : FH; therefore by conſtruction, the point M is in the line AC. Whence both K and M being in the line AC, determines the points of contact A and C; and FA drawn determines the point of contact B. Whence the five points A, B, E, C, D, are given.

PROP.

Fig.

PROP. LXXIV. *Prob.*

To deſcribe a conic ſection thro' a given point E, *to touch four right lines*, OT, TH, HR, RO, *given in poſition.* 91.

Draw the diagonals TR, HO, to interſect in K. Thro' K and the given point E, draw FQ; cutting the tangents in G, P, Q. Then find the point D ſo that $KG^2 : KP^2 :: DGE : DPE$; then D is another point of the curve. Therefore by the laſt Prop. draw the curve thro' the two points D, E; to touch the lines TO, TH, HR.

Or thus; find the point F, ſo that $FQ^2 : FG^2 :: EQD : EGD$. Then draw FH, and find the point M, ſo that FI : FH : : MI : MH. Thro' K and M draw CA; and thro' F and A, draw FB. Thro' B and K, draw BS. Then A, B, C, S are points of contact; thro' which and the point E, draw the ſection, by Prop. LXX.

For (Prop. LIV. Cor. B. I.) the lines joining the oppoſite points of contact, paſs thro' K. And ſince $KG^2 : KP^2 :: DGE : DPE$, (LIV. B. I.) D will be in the curve. And ſince FQ cuts the tangents TO, TH, and it is $FQ^2 : FG^2 :: EQD : EGD$; therefore (Prop. ib.) F is in the line joining the points of contact A, B. Alſo ſuppoſing M in the line AC, drawn thro' the points of contact of the tangents TO, HR; then (LIV. B. I.) $MI^2 : MH^2 :: LIN : LHN ::$ (for the tangents TG, TH) $FI^2 : FH^2$. And MI : MH : : FI : FH. On the contrary, ſuppoſing MI : MH : : FI : FH, then M will be in the line AC. Therefore a line drawn thro' M, K, will paſs thro' the points of contact A, C. And a line from F thro' A, paſſes thro' the point of contact B; and laſtly, a line drawn thro' B and K (LIV. Cor. B. I.), will paſs thro' the other point of contact S.

PROP.

Fig.

PROP. LXXV. *Prob.*

92. *To deſcribe a conic ſection, to touch five lines given in poſition,* GT, TH, HR, RO, OG.

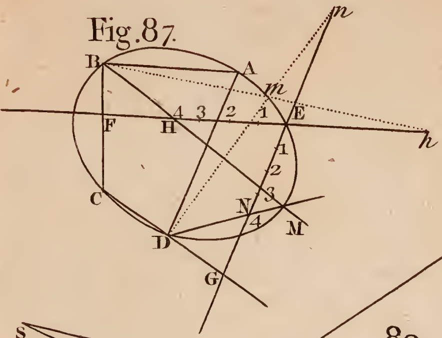
Take any four of the tangents forming a quadrilateral as IGTHI, leaving out the tangent OR; draw the diagonals IT, GH to interſect in L. Then leave out the tangent OG, and draw the diagonals of the quadrilateral KTHRK, to interſect in M; draw LM, to cut the tangents in A and C, the points of contact. Go thus round the figure; leaving out ſucceſſively the tangents GT, TH, HR; drawing the diagonals of the trapezia DHROD, ZROGZ, FOGTF; interſecting in N, P, Q. Then MN, NP, PQ, QL being drawn, will interſect the tangents in the points of contact, B and S, C and E, S and A, E and B. Then thro' the points A, B, C, S, E, deſcribe a conic ſection by Prop. LXX.

For (Prop. LIV. Cor. B. I.) the interſection of the diagonals of the trapezium IGTH, which is L. And alſo of the trapezium KTHR, which is M, are in the line AC, drawn thro' the points of contact A and C. Likewiſe the interſection of the diagonals of the trapezia KTHR, and DHRO; that is, M and N, are in the line BS, drawn thro' the points of contact B and S. Thoſe of DHRO and ZROG, N and P, are in CE. Thoſe of ZROG and FOGT, (P and Q), are in SA. Thoſe of FOGT and IGTH (Q and L), are in the line EB. Therefore A, B, C, S, E, are the five points of contact.

When the center or focus is given, it is equivalent to two lines or points; and an aſſymptote to an hyperbola, to two tangents.

Many

Fig. 87.



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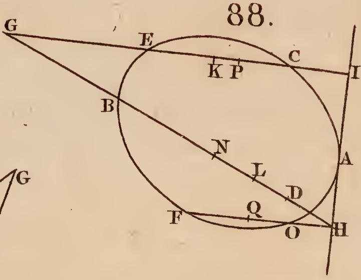

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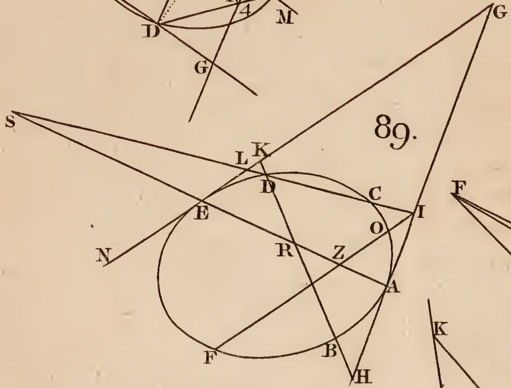

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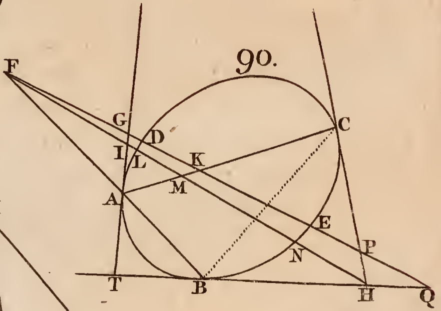

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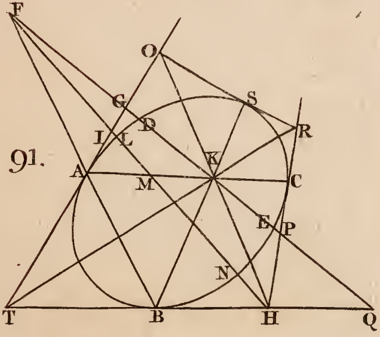

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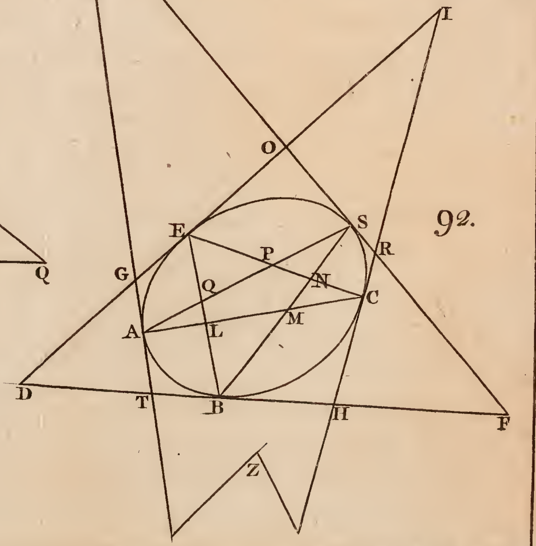


Pl. XXXII. *pa.* 222.

Many more problems might be added; but what Fig.
I have here given, are abundantly ſufficient for an 92.
introduction.

PROP. LXXVI.

If a cone ABC *be cut by a plane* LDG *parallel to* 93.
the ſide of the cone AB, *the ſection* LDG *will be a parabola.*

Draw the plane ED parallel to the baſe BGC. Then ſince AB is parallel to the plane LDG; a plane touching the cone in the line AB, will be parallel to LDG; whence arch BL = BG, and BI paſſing thro' the center of the baſe is perp. to LG, and LI = IG. And by the nature of the circle, the rectangle BIC = IL^2. From the vertex D draw DI. Then by ſimilar triangles AE : ED :: DI : IC = $\frac{ED \times DI}{AE}$; therefore BI × IC = IL^2, becomes $\frac{BI \times ED \times DI}{AE} = IL^2$, or $\frac{ED^2}{AE} \times DI = IL^2$. But AE and ED are given quantities; therefore make AE : ED :: ED : L; then $\frac{ED^2}{AE} = L$, and $L \times DI = IL^2$. Therefore (Prop. XVII.) the curve LDG is a parabola, and L the parameter.

Cor. 1. *The latus rectum* = $\frac{ED^2}{AE}$, *or a third proportional to* AE *and* ED. *And if* F *be the focus,* DF = $\frac{ED^2}{4AE}$.

By Prop. II.

Cor. 2. *Let* DP *be perp. to* AB, *then in a right cone,* 2PE = *the latus rectum*; *and* DF = ½ PE.

For if AN be perp. to ED, it biſſects it, and the triangles AEN, DEP are ſimilar; and AE : EN or

Fig. 93. or $\frac{1}{2}$ED : : ED : EP $= \frac{ED^2}{2AE}$, and 2EP $= \frac{ED^2}{AE}$ = latus rectum.

94. Cor. 3. *If* AF *be perp. to the baſe of the cone, and* AP *perp. to* DI *the axis of the parabola; and if* AN = AP, *and* HNG *drawn parallel to the baſe* BC; *then* HG *is equal to the latus rectum. And* HA *(on the oppoſite ſide)* = ED *the diameter at the vertex* D.

Draw AL parallel to BC, then the triangles AHN, APL are ſimilar and equal, and AH = AL = ED. And by ſimilar triangles (AED, AHG) AE : ED : : AH or ED : HG $= \frac{ED^2}{AE}$ = (Cor. 1.) latus rectum.

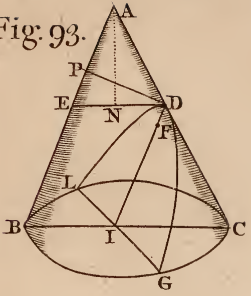
95. Cor. 4. *If* MVN *be a right cone*, F *the focus of the parabola* ARZ. *And if* AG *be made equal to* AF; *the circle* GI, *parallel to the baſe of the cone, will cut the plane of the parabola in the line* ED *the directrix of the parabola.*

For (Prop. I. Cor. 1.) if DE be the directrix, then AD = AF = (by conſtruction) AG. And ſince AN = AK, the triangles ANK, AGD are ſimilar, and DG parallel to NK; that is, GI parallel to MN, will cut the parabola in the line DE.

96. Cor. 5. *In a right cone* VMH, *if* F *be the focus, and* AG = AF, *and the circle* GNI *be drawn parallel to the baſe. Then the diſtance of any point* P *of the parabola, from the focus; that is* PF, *will be equal to* PN, *the diſtance of* P *from the periphery of the circle* GNI.

For let PQ be an ordinate, and draw QS parallel to the baſe of the cone; then ſince the plane SQP is parallel to the baſe, SG will be equal to PN, becauſe in a right cone the ſegments of the ſides contained between parallel circles are equal. And

Fig. 93.



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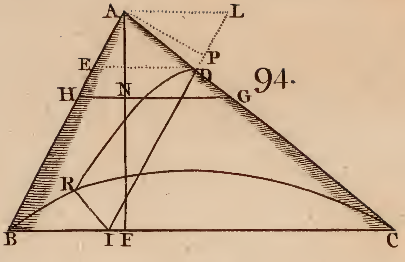

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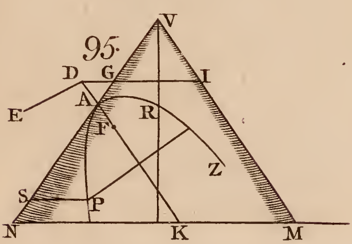

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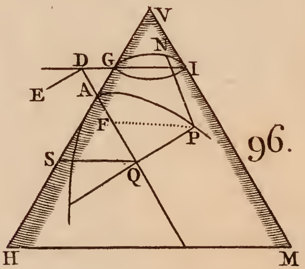


Pl.XXXIII. *the End.*

And ſince AQ = AS, and AD = AG, DE being the directrix; therefore DQ = SG. But (Prop. I.) FP = DQ = SG = PN. Fig. 96.

SCHOLIUM.

From this Prop. the reaſon appears, why the *Parabola* is called one of the conic ſections: and that is, becauſe a cone cut by a plane parallel to one of its ſides, will always form the curve called a parabola, as we have ſhewn in this propoſition. The properties of which curve we have demonſtrated at large in this Book; and of the other ſections in the foregoing books; and that from the mechanical deſcription thereof upon a plane, without regard to the cone or any other body whatever; without analytical calculations; without harmonical ratios; but by the pure ſyathetic method of demonſtration.

FINIS.

THE
NATURE AND PROPERTIES
OF
CURVE LINES.

BOOK I.

OF THE

Conchoid, Cissoid, Cycloid, Quadratrix, Logarithmic Curve; the Spiral of Archimedes, the Logarithmic Spiral, the Hyperbolic Spiral.

BOOK II.

Of Curve Lines in general, and their Affections.

Ex his principiis via ad majora Sternitur.
Newt. Quad. Curv.

THE

PREFACE.

THE enſuing Treatiſe, which is concerning curves of higher kinds, I have given here, as a proper ſequel to the Conic Sections. And to proceed gradually, I have in the firſt Book given account of many of the common ſort of curves, as the Conchoid, Ciſſoid, Cycloid, &c. and ſeveral ſorts of Spiral Lines; which will oblige theſe people that want to be acquainted with the nature of theſe curves. And as the reader will often meet with the mention of theſe curves, therefore he ought to know the nature of them, and their eſſential properties; and this is what every one ought to be acquainted with, that expect to make any great advances in this branch of ſcience. And in handling theſe, I have taken the ſhorteſt way of demonſtration I could find, whether geometrical or algebraical. But I have purpoſely omitted ſuch properties as could not be demonſtrated without having recourſe to fluxions. And for this reaſon alſo I have ſaid nothing about ſeveral other curves of like ſort.

In the ſecond Book I have treated of ſuch general properties, as belong to all curves univerſally. In Sect. 1, are laid down ſeveral theorems about the transformation of curves. By the help of ſome of theſe theorems, the area of one curve being given, or perhaps the length; the area of the other may be determined. By others, their lengths, ſurfaces, or ſolidities may be found. And this by common geometry without the help of fluxions.

In the ſecond ſection are laid down ſeveral propoſitions, relating to all ſorts of curves in general, their nature and properties. By theſe are determined the poſition of the axis, the ordinates, and their aſſymptotes, if they have any; and what curves have aſſymptotes, and what not. In this part I have given the learner a taſk of the higher geometry; and here is a large field, for any one that has a mind to exerciſe his invention therein.

In the third ſection you have ſeveral propoſitions concerning curve lines, drawn from fluxions, and which are ſerviceable for inveſtigating the nature of ſuch curves. When the reader is acquainted with all theſe things as an introduction, he may proceed with pleaſure to read other books on the ſame ſubject, and particularly that inimitable piece of Sir J. Newton's, entitled, An Enumeration of Lines of the third Order. *And therefore I hope, what is here delivered, will be grateful to all lovers of geometry, and particularly to ſuch as deſire to make any conſiderable progreſs therein.*

W. Emerſon.

BOOK I.

SECT. I.

Of the CONCHOID.

DEFINITIONS.

DEFIN. I.

Fig. 1. AQ is a ſtreight ruler in which the points A, B are fixed. Then if this ruler is ſo moved that it always paſſes thro' the fixed point P, whilſt the point B ſlides along the right line CD. The curve NAFG deſcribed by the point A, is called a *Conchoid*, or the *Conchoid of Nichomedes.*

DEF. II.

If B be between A and P, the curve AFG is called the *Superior Conchoid.*

DEF. III.

If *a* is between P and B, the curve *afg* is called the *Inferior Conchoid.*

DEF. IV.

2. If PT is perp. to CD, and if B*a* in the inferior conchoid, be greater than TP; the curve *a*P*f*ZP*g*, is called the *Nodated Conchoid.*

DEF. V.

1. The fixed point P is called the *Pole* or *Center* of the conchoid.

Fig. 1.

DEF. VI.

The line CD is called the *Directrix* of the conchoid.

PROP. I.

1. *All lines as* AB *comprehended between the curve and directrix, are equal, which paſs thro' the pole* P.

This is evident from the generation.

Cor. *If* BTF *be perp. to the directrix* CD, *any line* AB, *comprehended between the curve and directrix, and paſſing thro'* P, *is equal to the perpendicular* FT.

PROP. II.

1. *The directrix of the conchoid* CD *is an aſſymptote to the curve; or approaches infinitely near, but never touches it.*

From any point A of the curve, draw AP to the pole, and AH perp. to CT. Then the triangles PBT, ABH are ſimilar, and BP : PT : : AB : $AH = \frac{AB \times PT}{PB}$. Then ſince the diſtance of the point A from CD; that is, AH is always reciprocally as PB; AH continually diminiſhes as PB increaſes, and when PB is infinitely great, AH will be infinitely ſmall. Therefore CD is an aſſymptote to the curve.

And the ſame reaſoning may be applied to the inferior conchoid *afg*.

Fig.

PROP. III.

If from any point A *of the curve,* AH, FTP *be drawn perpend. to the directrix* HT; *and* AL *parallel to it. And if the quadrant* FGK *be deſcribed cutting* AL *in* G. *Then* PL × LG = TL × LA. 3.

For the triangles AHB, TLG, are ſimilar and equal; for the angles at H and L are right. And AB = TF = TG, and AH = TL; therefore BH = GL. Alſo the triangles AHB, ALP, are ſimilar; whence AH : HB : : PL : AL; that is, TL : GL : : PL : AL; therefore PL × LG = TL × AL.

Cor. 1. *The ſame things ſuppoſed,* AG = BT, *and* GL = HB.
For AL = HT, and GL = HB; therefore by ſubtraction, AG = BT.

Cor. 2. *This Prop. holds good alſo in the inferior conchoid.* 4.
This appears by applying the ſame demonſtration to the inferior conchoid, denoted by ſmall letters.

Cor. *In the inferior conchoid,* $gl = hb$.

PROP. IV.

In the ſuperior conchoid, if PTF *and* AH *be perp. to the directrix* HT, *and* AL *parallel to it.* $TL^2 : TF^2 - AH^2 : : PL^2 : AL^2$. 3.

For by Prop. III. TL : GL : : PL : AL; and $TL^2 : GL^2 : : PL^2 : AL^2$. But $GL^2 = GT^2 - TL^2 = TF^2 - AH^2$; therefore $TL^2 : TF^2 - AH^2 : : PL^2 : AL^2$.

T 4 Cor.

Fig. 3. Cor. *Hence putting* FT $= a$, PT $= b$, *axis* TH $= x$, *ordinate* HA $= y$; *then the equation of the ſuperior conchoid is* $\frac{\overline{b+y}\sqrt{aa-yy}}{y} = x =$ AL, *or being reduced* $y^4 + 2by^3 + \overline{bb - aa + xx} \times yy - 2aaby = aabb$.

PROP. V.

4. *In the inferior conchoid, if pft and ah be perp. to the directrix bt, and al parallel to it. Then* $tl^2 : tf^2 - ah^2 :: Pl^2 : al^2$.

For (Pr. III. Cor. 2.) $tl : gl :: Pl : al$. And $tl^2 : gl^2 :: Pl^2 : al^2$. But $gl^2 = bh^2 = ba^2 - ah^2 = tf^2 - ah^2$; therefore $tl^2 : tf^2 - ah^2 :: Pl^2 : al^2$.

Cor. 1. *Hence putting* $tf = a$, tP $= b$, *axis* $th = x$, *ordinate* $ha = y$, *the equation of the inferior conchoid will be* $\frac{\overline{b-y}\sqrt{aa-yy}}{y} = x = al$. *And when reduced,* $y^4 - 2by^3 + \overline{bb - aa + xx} \times yy + 2aaby = aabb$.

Cor. 2. *The ſame equation holds for the nodated conchoid, all the difference being, that a is here greater than b.*

SECT.

SECT. II.

Of the CISSOID.

DEFINITIONS.

DEF. I.

LET ABG be a circle, C the center, QED a ſquare; alſo P, F two fixed points in the lines AP, ED, ſo that GP = radius CG = EF = FD. Alſo let CD produced ad infinitum, be perp. to CP. Let the ſquare QED be ſo moved, that the ſide EQ may always paſs thro' the point P, whilſt the end D ſlides along the right line CD. Then the point F (the middle of ED) will deſcribe the curve GBF, called the *Ciſſoid*, or the *Ciſſoid of Diocles*. Fig. 5.

DEF. II.

The circle ABG is called the *Generating Circle*.

DEF. III.

The line CD is called the *Directrix*.

PROP. I.

If AB *and* BG *be quadrants, the ciſſoid paſſes thro' the point* B. 5.

For when the ſquare PED comes into the poſition PCB, then the angles C and E being right, EF falls upon CB, and F upon B, cutting off the quadrant GB.

PROP. II.

6. *If from any point* F *of the curve,* FL *be drawn parallel to the directrix* CD, *to cut the circle in* M. *Then* LM, LG, LF, *are in geometrical progreſſion.*

For draw GH parallel to CD, and FH parallel to AG. Then the triangles CPR, ERD are ſimilar and equal. For the oppoſite angles at R are equal, the angles at C and E are right, and CP $=$ ED; therefore CR $=$ ER, and conſequently ER $+$ RD $=$ CD. Let LF or GH $= x$, LG or FH $= y$, AC $= a$. Then FO $= y - a$, OD $= \sqrt{FD^2 - FO^2}$ $= \sqrt{aa - \overline{y - a}^2} = \sqrt{aa - yy + 2ay - aa} =$ $\sqrt{2ay - yy}$. And OF $+$ FD $=$ OF $+$ OH $= y$. The right angled triangles FOD, ERD are ſimilar, whence OD : $\left\{\begin{matrix}\text{OF}\\ \text{FD}\end{matrix}\right.$ :: ED : $\left\{\begin{matrix}\text{ER}\\ \text{RD}\end{matrix}\right.$. And by compounding, OD : OF $+$ FD :: ED : ER $+$ RD or CD; that is, $\sqrt{2ay - yy} : y :: 2a : x +$ OD or $x + \sqrt{2ay - yy}$; therefore $x\sqrt{2ay - yy} + 2ay - yy = 2ay$, and $x\sqrt{2ay - yy} = yy$, but by nature of the circle, LM $= \sqrt{2ay - yy}$, therefore LM $(\sqrt{2ay - yy})$: LG (y) :: LG (y) : LF (x).

Cor. 1. *Hence* OD $=$ LM, *the ordinate of the circle.*

For OD was $= \sqrt{2ay - yy}$, and LM $=$ $\sqrt{2ay - yy}$.

Cor. 2. *If* GH $= x$, HF $= y$, CG $= a$, *the equation of the curve is* $2axx - yxx = y^3$.

For $x\sqrt{2ay - yy} = yy$, and by ſquaring, $2ayxx - yyxx = y^4$; and dividing, $2axx - yxx = y^3$.

Cor. 3. *The four lines* AL, LM, LG, LF, *are* Fig.
in continued proportion. 6.

For by the nature of the circle AL : LM : : LM : LG : : (by this Prop.) LG : LF.

Cor. 4. *Produce* FL *to* N, *and draw* NC, *then will* CD = FN.

For LF = CO, and NL = LM = (Cor. 1.) OD, therefore NF = CD.

Cor. 5. $FH : AL :: GH^2 : FH^2$.

For $y : 2a - y :: xx : yy$, by Cor. 2.

PROP. III.

If thro' any point of the curve F, LF *be drawn* 7.
perp. to the diameter AG, *cutting the circle in* M; *and the line* GFV *be drawn, cutting the circle in* V; *then arch* AV = *arch* GM.

For (Prop. II.) LM : LG : : LG : LF : : (ſimilar triangles) GZ : ZV. And $LM^2 : LG^2 :: GZ^2 : ZV^2$; that is, $ALG : LG^2 :: GZ^2 : AZG$. And dividing the antecedents by LG, and the conſequents by GZ, then AL : LG : : GZ : AZ. And compounding, AL + LG : LG : : GZ + AZ : AZ; that is, AG : LG : : AG : AZ; therefore LG = AZ, and LM = ZV, and arch GM = AV.

Cor. 1. *Arch* AM = *arch* GV.

Cor. 2. *If two lines* AM, GV *move about the points* A, G, *ſo that their interſection be always in the directrix* CB; *and if* ML *be perp. to* AG; *then the interſection of* GV *and* ML, *at* F, *will be in the ciſſoid.*

For by that means the arches AV and MG will be equal; and F will be in the curve by this Prop.

Cor. 3. *The lines* AO, GO, DO *are all equal.*

For

Fig. 7. For the triangle AOG is iſoceles, and ſince AC = CG, therefore AO = OD.

Cor. 4. *The line* AM = ID = GV.
For AO = OD, and by reaſon of the equal arches VB, BM, IO is = OM; and ID = AM or GV.

Cor. 5. *The lines* OI, OV, OF, OM *are equal.*
Becauſe the arches VB, BM, and the lines ZC, CL, are equal.

Cor. 6. *The line* GF = AI *or* MD.
For the triangles AZI, GLF are ſimilar and equal.

PROP. IV.

8. *If* GS *be perp. to the diameter* AG, *and from any point* F *in the curve*, FN *be drawn parallel to* GS, *cutting the circle in* N, *and* ANS *be drawn. Then* NF = GS.

Draw GFV, and produce NF to M. Then ſince arch NG = arch GM = (Prop. III.) arch AV; therefore the angle NAG = AGV; whence AS is parallel to GV. And ſince NF is parallel to SG; therefore NFGS is a parallelogram, and NF = GS.

Cor. 1. *The line* IZ = LF.
For the triangles AZI and GLF are ſimilar and equal.

Cor. 2. *If* NF *be drawn perp. to* AG, *thro' any point* F *in the curve, and* GF *be drawn. Then* NF, GF, LF, *are continually proportional.*
For draw NG, then ſince arch NG = arch AV, add GV, then NGV = GVA = a ſemicircle; whence NGV or NGF is a right angle. Therefore in the right angled triangle NGF, where GL is perp. to NF, it is NF : GF : : GF : LF.

Cor. 3. GS, GF, LF, *are continually proportional.* Fig. 8.

PROP. V.

If AE *be perpendicular to* AG *the diameter of the generating circle; then* AE *is an aſſymptote to the curve.* 9.

For let GH $= x$, HF $= y$, AG $= 2a$. Then (Cor. 2. Prop. II.) the equation of the curve is $2axx - yxx = y^3$, and $xx = \frac{y^3}{2a - y}$, where EF $=$ $2a - y$. So that as xx continually increaſes, and y^3 cannot exceed $\overline{2a}^3$; therefore $2a - y$ or EF continually decreaſes; and when xx becomes infinitely great, $2a - y$ or EF becomes infinitely ſmall; and when $y = 2a$, then $xx = \frac{\overline{2a}^3}{0}$ $=$ infinitely. Therefore AE is an aſſymptote.

PROP. VI.

Thro' any point F *of the ciſſoid, draw* GH, *cutting the circle in* D, *and the aſſymptote in* H. *Then* GF $=$ DH. 10

Thro' D and F draw BD, LM parallel to AH. Then (Prop. III.) arch AD $=$ arch GM. Whence GL $=$ AB, and by reaſon of the parallels AH, BD, LF; GF $=$ DH.

Cor. *The line* GD $=$ *line* FH; *or the line within the circle is* $=$ *the line without the ciſſoid.*

PROP.

Fig.

PROP. VII.

8. *Thro' any point* F, *draw* NF *perp. to* AG, *and* GS *parallel to it, alſo draw* ANS. *Then* GN : GF : : GL : LF : : AG : GS.

For it is ſhewn in Cor. 2. Prop. IV. that NGF is a right angled triangle, and GL perp to NF. Therefore GN : GF : : GL : LF : : (by ſimilar triangles) AZ : ZI : : AG : GS.

Cor. AN : NG : : NG : GF : : GL : LF. Still by ſimilar triangles.

PROP. VIII.

8. *Thro' any point* F, *draw* NF *perp. to* AG, *and* GS *parallel to it, alſo draw* ANS. *Then* GA : GL : : GS^2 : GF^2.

For (Prop. IV. Cor. 3.) GS : GF : : GF : LF; and GS^2 : GF^2 : : GS : LF : : (ſimilar triangles) AG : GL.

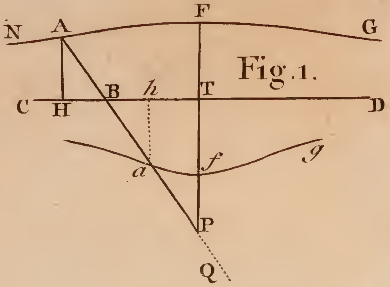
PROP. IX.

All ciſſoids are ſimilar figures.

This is plain, becauſe the abſciſſæ and ordinates of ſeveral ciſſoids, will be in the ſame ratio; when either of them is in a given ratio, to the diameter of its generating circle.

SECT.

Fig. 1.



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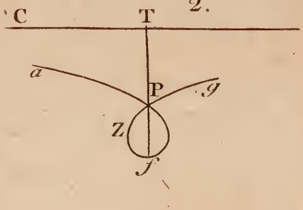

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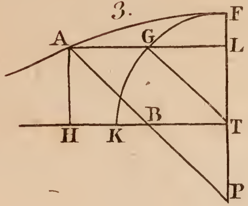

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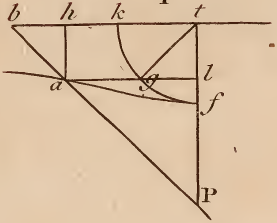

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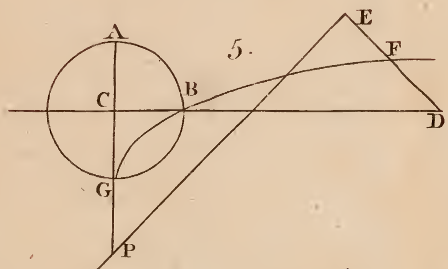

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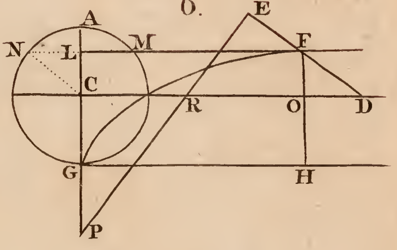

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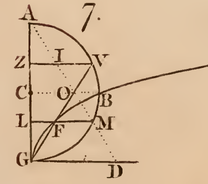

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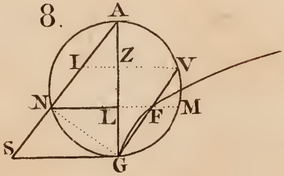

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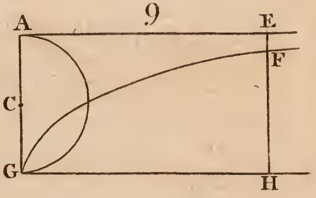

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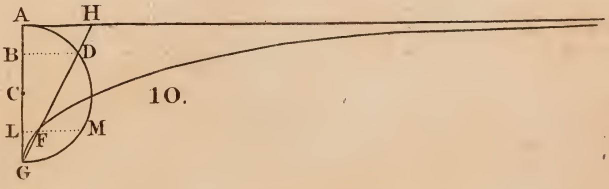


SECT. III.

Of the CYCLOID.

DEFINITIONS.

DEF. I.

IF a circle DEBL be made to roll along the right Fig.
line AF, until the fixed point D, which at firſt 11.
touched the line at A, comes to touch it again at F; that point D will deſcribe the curve ADGF, called a *Cycloid or Trochoid.*

DEF. II.

The circle DEBL is called the *Generating Circle.*

DEF. III.

The line DB perpendicular to the middle of AF, is the *Axis* of the cycloid.

PROP. I.

The baſe of the cycloid AF *is equal to the circum-* 11.
ference DEBL *of the generating circle.*

For all the particles of the circle are ſucceſſively applied to the right line AF; and therefore the ſum of all, or the whole circle is equal to that line.

PROP. II.

Draw PM *parallel to the baſe* BF, *then the line* 12.
PM *is equal to the arch* DP.

For let the circle DPE come to the place LME; then by the circular motion about C, the deſcribing point

Fig. 12. point D will have moved thro' the arch DP. And by the progreſſive motion along BF, the point H is come to E, ſo that BE = arch BH, which is equal the arch DP or LM. But EM is parallel to BP, becauſe the arch LM = DP. Whence BPME is a parallelogram. And PM = BE = arch BH = arch DP.

Cor. *The ordinate of the cycloid is equal to the ſum of the arch and its ſine.* AM = DP + AP.

PROP. III.

13. *If* AM *be an ordinate cutting the circle in* P, MT *a tangent at* M. MT *is parallel to the cord* PD.

For whilſt the cord BP or EM is deſcribing the infinitely ſmall part of the curve at M, the point E may be conſidered as fixed for a moment, and therefore the ſmall arch at M will be perp. to EM. But by the demonſtration of the laſt Prop. BPME is a parallelogram, and EM parallel to BP. But the angle DPB in a ſemicircle is alſo right; therefore DP is parallel to the particle of the curve at M, or to the tangent MT.

Cor. 1. *If* ME *be parallel to* PB, *then* ME *is perp. to the curve in* M.

Cor. 2. *If* MT *be a tangent, and the cord* DP *be drawn, then* AP : AD : : AM : AT.

For DP, TM being parallel, the triangles ADP, ATM are ſimilar.

PROP. IV.

14. *If the ordinate* GDM *be drawn; the length of the arch of the cycloid* AM *is double the correſpondent cord* AD, *in the generating circle.*

Draw the ordinate E*np* infinitely near GDM, draw the cord A*on*, and D*r* perp. to it. Then the

Fig. 14. the triangle oDn is iſoceles; and An biſſects the angle DnE. For the angle $AnD = AEn = EnA = noD$, by reaſon of the parallels; that is, $onD = noD$; and $nD = oD$, whence $nr = \frac{1}{2} no$ But (Prop. III.) the tangent pT is parallel to the cord An; therefore $Monp$ is a parallelogram, and $Mp = no = 2nr$; that is, the difference of the arches Ap, AM, is twice the difference of the cords An, AD. And as this always happens till they vaniſh at A; therefore the whole arch Ap is twice the cord An, and the arch $AM =$ twice the cord AD.

Cor. 1. *The length of the ſemicycloid* AMF *is double the diameter* AB, *of the generating circle.*

Cor. 2. *The length of the whole cycloidal curve is four times the diameter of the generating circle.*

Cor. 3. *If* GM *is an ordinate, then* AG, *arch* AM, *and* 4AB, *are in continual proportion.*

For $AM = 2AD$, and $AM^2 = 4AD^2 = 4AG \times AB$, and $AG : AM :: AM : 4AB$.

Cor. 4. AG, *arch* AM, *and* 2AF *are in continual proportion.*

For (Cor. 2.) $4AB = 2AF$.

PROP. V.

15. *If* GM *be an ordinate,* AG *the abſciſſa; then* $AB : AG :: AF^2 : AM^2$.

For (Prop. IV. and Cor. 1.) $AF = 2AB$, and $AM = 2AD$; therefore $AF^2 = 4AB^2$, and $AM^2 = 4AD^2$. And $AF^2 : AM^2 :: 4AB^2 : 4AD^2 :: AB^2 : AD^2 :: AB : AG$.

Cor. *The abſciſſa* AG *is directly as the ſquare of the arch* AM^2.

Fig. 15. For $AF^2 : AM^2 :: AB : AG :: 4AB^2 : 4AB \times AG ::$ (IV. Cor. 1.) $AF^2 : 4AB \times AG$. Therefore $AM^2 = 4AB \times AG$. And therefore AM^2 is as AG. This alſo might be inferred from Cor. 3. Prop. IV.

PROP. VI.

16. *If* ME *be perp. to the baſe* BF; *then* $AB \times ME = AB \times MF - \frac{1}{4} MF^2$.

For (IV. Cor. 3.) $4AB \times AG = AM^2$. But $AG = AB - BG$, and $AM = AF - MF$; therefore $4AB \times \overline{AB - BG} = \overline{AF - FM}^2 =$ (IV. Cor. 1.) $\overline{2AB - FM}^2$; that is, $4AB^2 - 4AB \times BG = 4AB^2 - 4AB \times FM + FM^2$. And $4AB \times FM - 4AB \times BG = FM^2$. Whence $AB \times FM = AB \times BG + \frac{1}{4} FM^2$; or $AB \times BG = AB \times FM - \frac{1}{4} FM^2$; that is, $AB \times ME = AB \times MF - \frac{1}{4} MF^2$.

PROP. VII.

16. *If* ME *be perp. to the baſe* BF, MG *parallel to it*; *then* EF = *arch* BD — *the ſine* GD.

For GM or BE = (II. Cor.) AD + DG, and EF = BF — BE = (Pr. I.) ADB — BE = ADB — AD — DG = DB — DG.

PROP. VIII.

17. *If* GDM *be an ordinate*, AN *parallel to it*, *and* NM *parallel to* AB; *the external cycloidal ſpace* ANM *is equal to the circular ſegment* AGD.

For draw the tangent MT and cord AD, and *mnp* parallel to NM. Then (Prop. III. Cor. 2.)

AG : GD : : TG : GM : : (ſimilar triangles) pm : Mm; therefore AG × Mm = GD × pm; that is, the area $NMpn$ = area $GgdD$. But theſe are the correſponding elementa of the cycloid and circle; therefore by the method of indiviſibles, the ſum of all the $NMpn$, or the area ANM = ſum of all the $GgdD$ or the area AGD, of the circle. Fig. 17.

Cor. 1. *The area of the internal cycloidal ſpace is equal to the rectangle* AG × GM — *the circular area* AGD.

For area AGM = AG × GM — ANM = AG × GM — area AGD.

Cor. 2. *The area of the whole cycloidal ſpace* AMFB *is equal to thrice the area of the ſemicircle* ADB.

For (Cor. 1.) area AMFB = rectangle AB × BF — area ADB = (Prop. I.) AB × ſemicircumference ADB — area ADB, = 4 areas ADB — area ADB = 3 times the area ADB.

Cor. 3. *The whole cycloidal ſpace* AMFB *is* $\frac{3}{4}$ *the circumſcribing parallelogram.*

For the external ſpace is equal to the ſemicircle ADB = $\frac{1}{4}$ AB × arch ADB = $\frac{1}{4}$ AB × BF.

Cor. 4. *The whole cycloidal ſpace without the circle,* ADBFM *is double the ſemicircle* ADB.

SECT: IV.

Of the QUADRATRIX.

DEFINITIONS.

DEF. I.

Fig. 18. ACBL is the quadrant of a circle, and ſuppoſe the radius CL to move uniformly round the center C, from A to B; and at the ſame time, the line FG to move uniformly along AC, from A to C, continuing always parallel to CB; and ſo that both motions begin and end together, at A and B. Then the interſection N of the lines CL, FG, during the motion, will deſcribe the curve AND, called the *Quadratrix, or Dinoſtratus's Quadratrix.*

DEF. II.

The circle ALB is called the *Generating Circle.*

DEF. III.

The radius AC is called the *Axis*, and CD the *Baſe.*

PROP. I.

18. *Thro' any point of the curve* N, *draw the radius* CNL, *and* NF *parallel to* CB. *Then* AF : AC : : *angle* ACN : *angle* ACD.

For by the generation, AF : AC : : arch AL : arch AB : : angle ACL : angle ACB.

Cor. 1. AF : FC : : *angle* ACN : *angle* NCD.

Cor. 2. *The abſciſſæ* AF, *are as the angles at the center* ACN, *or as the arches which meaſure them.*

PROP.

Fig.

PROP. II.

The baſe is a third proportional to the quadrant and the axis. ALB : AC : : AC : CD. 19.

Thro' N infinitely near D, draw CL; and draw NF parallel to CD; and NI, LS perp. to it. Then (Prop. I.) AC : AF : : angle ACB : angle ACL : : arch ALB : arch AL. And by diviſion, AC : CF or NI : : arch ALB : arch LB. And alternately, ALB : AC or CB : : LB : NI. But when L coincides with B, S alſo coincides with B; and N and I, with D. Therefore ALB : CB : : LB or LS : NI : : (ſimilar triangles) CS or CB : CI or CD. That is, ALB : CB or CA : : CB or CA : CD.

PROP. III.

With the radius equal to the baſe CD *deſcribe the quadrant* DNH. *Then will this quadrant* DNH *be equal to the axis* AC. 20.

Deſcribe the quadrant ALB; then (Prop. II.) ALB : AC : : AC : CD : : (ſimilar ſectors) ALB : DNH; therefore AC = DNH.

PROP. IV.

Thro' any point N *of the curve, draw the radius* CL; *and* FN *parallel to the baſe*; *then* CD : CA : : AF : *arch* AL. 19.

For (Prop. II.) CD : CA : : CA : ALB : : (Prop. I. Cor. 2.) AF : arch AL.

Fig.

PROP. V.

21. *Upon the baſe* CD, *deſcribe the quadrant* DPH. *Thro' any point* N *of the curve draw* NF *perp. to* AC, *and* CN *cutting the inſcribed quadrant in* P. *Then* AF = *arch* HP.

Draw the quadrant ALB; then (Prop. IV.) AF : arch AL : : CD : CA : : (ſimilar ſectors) arch HP : arch AL. Therefore AF = arch HP.

Cor. FC = *arch* PD.
For (Prop. III.) AC = HPD, and AF = HP; therefore FC = PD.

PROP. VI.

22. *If the radius* CA *be continued to* Q, *ſo that* AQ = AC; *and the ſemicircle* BAK *be deſcribed. And make* AQ : AF : : AK : AL. *And draw* CL, *and* FN *parallel to* CK, *cutting* CL *in* N; *then will* N *be in the curve of the quadratrix continued without the circle.*

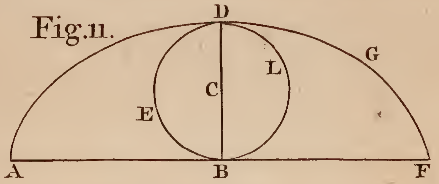
For the part AN is conſtructed by the ſame law as the part AD within the circle.

Cor. 1. *The quadratrix* DAN*n is continued ad infinitum, by finding a ſufficient number of points* N, *n*, *&c. by the ſame rule.*

Cor. 2. *The line* QX, *erected perpendicular to the line* QAC, *is an aſſymptote to the curve.*

For ſince AF : AL : : AQ : AK, when F is infinitely near Q, L will be infinitely near K, and the interſection N will be infinitely diſtant. Whence the curve AN, and the line QX, continually approach to one another; and yet never meet, but at an infinite diſtance.

Fig. 11.



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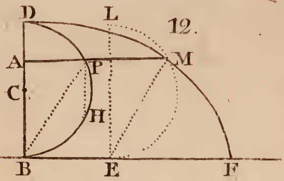

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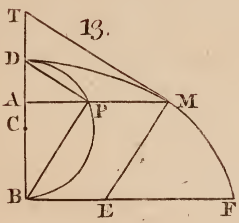

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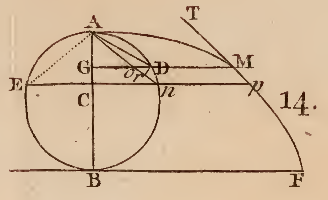

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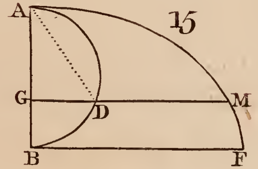

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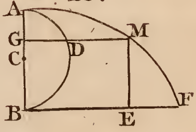

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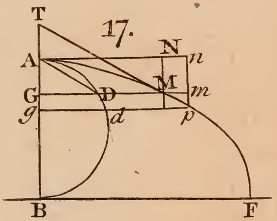

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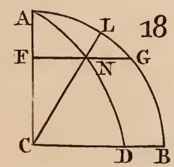

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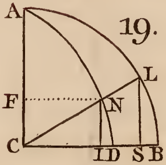

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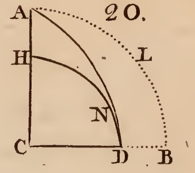

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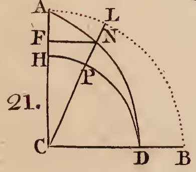

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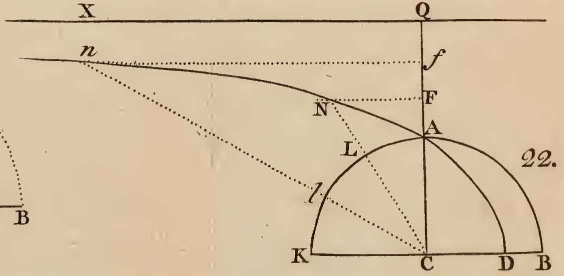


SECT. V.

Of the LOGARITHMIC CURVE.

DEFINITIONS.

DEF. I.

IF in the right line AE, the parts AB, AC, AD, AE, &c. be taken in arithmetic progreſſion, from the fixed point A; and at the points A, B, C, D, E, &c. the ordinates AF, BG, CH, DI, EK, &c. be drawn parallel to one another, and in geometrical progreſſion. And if theſe ordinates are ſuppoſed infinite in number and infinitely near to one another; then the curve line FGHIK drawn thro' the extremities of all the ordinates, is called the *Logarithmic Curve.* Fig. 23.

DEF. II.

The line AE is called the *Axis* of the curve.

PROP. I.

The axis AE *is an aſſymptote to the curve.* 24.

Let the curve be continued backwards from A, and the ordinates AF, *bg*, *ch*, &c. continually decreaſing, and let A*e* $= n \times$ AB. Then ſince AF, *bg*, *ch*, &c. are continual proportionals; therefore it will be as AF : *ek* : : $AF^n : bg^n$. Now as n increaſes, the ratio of AF^n to bg^n, and that of AF to *ek*, will continually increaſe; and ſince AF is given, *ek* continually decreaſes. And when n is infinite, AF^n will be infinitely greater than bg^n, and conſequently *ek* infinitely leſs than AF; therefore A*e* is an aſſymptote.

Fig.

PROP. II.

25. *If* TE *is a tangent to the curve at* E, BE *an ordinate: then the ſubtangent* TB *is every where the ſame, wherever the point* E *is taken.*

Suppoſe the ordinates BE, CF, DG equidiſtant and infinitely near each other; draw the tangent FV, and draw E*n*, F*r*, parallel to AD. Then ſince BC = CD, it will be BE : CF : : CF : DG : : (by diviſion) CF — BE : DG — CF : : *n*F : *r*G. And alternately BE : *n*F : : CF : *r*G. But the triangles F*n*E and EBT are ſimilar, as alſo the triangles G*r*F, and FCV. Therefore BT : *n*E : : BE : *n*F : : CF : *r*G : : CV : *r*F, but *n*E = *r*F; therefore BT = CV, and ſo it will be every where.

PROP. III.

26. *If* AF, BG, DI, EK, *be four ordinates, and* AB *the diſtance of two of them be equal to* DE, *the diſtance of the other two; and thro' the points* G, F, *and* K, I, *the lines* GFT, KIV *be drawn, cutting the axis in* T *and* V. *Then will* AT = DV.

Draw FS, IL parallel to AE. Then ſince AB = DE, therefore AD = BE, and AF, DI, BG, EK are four terms in the ſcale of continual proportionals, of which the firſt two, and the laſt two are equidiſtant; and therefore it will be AF : DI : : BG : EK. And AF : BG : : DI : EK; and AF : BG — AF : : DI : EK — DI, or AF : GS : : DI : LK. But the triangles TAF, FSG, are ſimilar, as alſo the triangles VDI, ILK. Therefore TA : FS : : AF : SG : : DI : LK : : VD : IL, but FS = IL, therefore TA = VD.

PROP.

PROP. IV.

The ſpace BELG, *comprehended between any two ordinates* EL, BG *is equal to* TE × SL, *the rectangle of the ſubtangent and the difference of the ordinates.* 27.

For let DI be infinitely near EL, and draw I*r* parallel to AE. Then LT being a tangent, the triangles LTE and LI*r* are ſimilar, whence LE : ET : : L*r* : *r*I; therefore LE × *r*I = ET × L*r*. But LE × *r*I = area DELI, therefore ET × L*r* = area DELI. But (Prop. II.) ET is a given quantity, whence the ſum of all the DELI or the area BELG = ET × ſum of all the L*r*, or ET × LS.

Cor. 1. *The area of the whole curve infinitely produced towards* A, *is equal to the rectangle of the ſubtangent and ordinate,* LE × ET.
For then LS becomes equal to LE, by Prop. I.

Cor. 2. *The whole ſpace infinite towards* A, *is double the triangle* TEL.

Cor. 3. *The logarithetic ſpaces infinitely long, are as the bounding ordinates.*

Cor. 4. *The ſpace between any two ordinates, is to the ſpace between any other two; as the difference of the firſt two; to the difference of the laſt two.*

PROP. V.

The ſolid made by the infinitely long ſpace FLEA *revolving about the axis* AE, *is equal to half a cylinder of the ſame baſe, and hight equal to the ſubtangent;* 27.
$\frac{TE}{2}$ × *circle* EL.

For let EL, DI, be two ordinates infinitely near, draw I*r* parallel to AE. Then the line LT being a tan-

Fig. 27. a tangent, the triangles TEL, IrL are ſimilar; therefore $LE : ET :: Lr : rI$, and $LE \times rI = Lr \times ET$, and $LE^2 \times rI = LE \times Lr \times ET$. Put $LE = y$, $ET = t$, $c = 3.1416$; then $rI \times yy = Lr \times ty$, and $rI \times cyy = Lr \times cty$; but $rI \times cyy$ = ſolid LIDE; whence $Lr \times cty$ = ſolid LIDE. Therefore if LE be divided into an infinite number of equal parts equal to $Lr = 1$, then the ſum of all the cty = ſum of all the ſolids LIDE. But (Arithm. Inf. Prop. II.) the ſum of all the y is $= \frac{yy}{2}$, and the ſum of all the $cty = \frac{ctyy}{2}$ = ſum of all the ſolids LIDE = the whole ſolid LEAF infinitely extended towards A. But cyy = area of the circle whoſe radius is EL or y; therefore the whole ſolid LEAF $= \frac{1}{2} t \times$ area of the circle whoſe radius is LE.

Cor. 1. *The infinitely long ſolid is to a cone of the ſame baſe, and hight the ſubtangent; as 3 to 2.*

Cor. 2. *The ſolid* LEBG, *contained between two ordinates, is half the cylinder, whoſe hight is the ſubtangent, and baſe the difference of the circles whoſe radii are* LE, GB.

For the ſolid LEAF $= \frac{1}{2} t \times$ circle LE, and GBAF $= \frac{1}{2} t \times$ circle GB; therefore the difference LEBG $= \frac{1}{2} t \times :$ circle LE — circle GB.

PROP. VI.

23. *If* $AF = 1$, $BG = a$, $AE = x \times AB$, $EK = y$; *then the equation of the curve is* $y = a^x$.

For let AB, AC, AD, AE, &c. be arithmetic proportionals, then AF, BG, CH, &c. are in geometrical progreſſion. And we ſhall have $AF^x : BG^x :: AF : EK$; that is, $1 : BG^x :: 1 : EK$, or $1 : a^x :: 1 : y$, and $y = a^x$.

PROP.

PROP. VII.

If the ordinate AF, BG, CH, *&c. repreſent num-* 23.
bers in geometrical progreſſion, and be placed at equal diſtances; the abſciſſas AB, AC, AD, *&c. being in arithmetic progreſſion, will repreſent the logarithms of theſe numbers.*

That the abſciſſas are in arithmetical progreſſion, when the ordinates are in geometrial progreſſion, appears from Def. 1. And from the nature of logarithms, if they be in arithmetical progreſſion, the numbers anſwering will be in geometrical progreſſion. Therefore the abſciſſas in this curve are the logarithms of the ordinates belonging thereto.

Cor. *Hence logarithms may be of as many different forms as we pleaſe. But as a multiplicity of ways would be uſeleſs, Mathematicians confine themſelves to two ſorts only.* 1. *When* AF, BG, CH, DI, *&c. are* 1, 10, 100, *&c. and* AB, AC, AD, *&c. are* 1, 2, 3, *&c. and theſe are called Brigs, or the common logarithms, and the ſubtangent is,* 43429448 *&c.* 2. *Napier's or the hyperbolic logarithms, where the ſubtangent is* 1.

PROP. VIII.

If FG, QS *be two logarithmic curves,* $AF =$ 28.
$PQ = 1$, $BG = HS =$ *any other ordinate*; GT, 29.
SV, *tangents*; *then will* AB : PH : : TB : VH.

Let the equal ordinates LI, NK, be placed infinitely near BG, HS; then ſince they are equal, they have a like poſition upon their abſciſſas; that is, LB : NH : : AB : PH. Draw I*n*, K*r*, parallel to AB, PH. Then the triangles GI*n*, GTB, as alſo SK*r*, SVH, are ſimilar; whence BT : *n*I : : BG :

Fig. $BG : nG :: HS : rS :: HV : rK$; and alternately,
28. $BT : HV :: nI$ or $LB : rK$ or $NH :: AB : PH$.
29.

Cor. 1. *The logarithms of equal numbers are proportional to the ſubtangents of the curves they belong to.* For AB, PH are the logarithms of the equal numbers BG, HS.

Cor. 2. *In a number extremely near* 1; *its exceſs above* 1, *is to its logarithm*; *as* 1, *to the ſubtangent.*
28. For let LI be $= 1$, and BG infinitely near it; then by ſimilar triangles, $Gn : nI$ or LB its log. :: IL or 1 : LT the ſubtangent.

Cor. 3. *The diſtances of two equal ordinates in two different curves, are as the ſubtangents of the curves.*

PROP. IX.

23. *If* DI, EK *be two ordinates*; *then* $DE = log. : \frac{EK}{DI}$.

For $AE = \log. : EK$, and $AD = \log. : DI$, whence $AE - AD = \log. : EK - \log. : DI =$ (by the nature of logarithms) $\log. : \frac{EK}{DI}$; that is, $DE = \log. : \frac{EK}{DI}$, for that curve.

Cor. *If* BG, CH, *and* DI, EK, *be any ordinates*; *then* $BC : DE :: log. : \frac{CH}{BG} : log. : \frac{EK}{DI}$, *by any table of logarithms.*

For in this fig. $BC = \log. : \frac{CH}{BG}$, and $DE = \log. : \frac{EK}{DI}$; and (VIII. Cor. 1.) $BC : DE :: \log. : \frac{CH}{BG} : \log : \frac{EK}{DI}$, by any table.

SECT.

SECT. VI.

Of the SPIRAL *of* Archimedes.

DEFINITIONS.

DEF. I.

IF the radius CA be moved uniformly round the center C, deſcribing the circle ABDA; and at the ſame time a point is ſuppoſed to move uniformly along that radius from C to A, ſo that both motions begin and end together; the curve CFGA deſcribed by that point, is called the *Spiral of Archimedes*. Fig. 30.

And if the radius (produced) move round a ſecond time, while the deſcribing point continues its motion, the curve deſcribed, is the *Second Spiral*; and ſo on.

DEF. II.

The circle ABDA is called the *Generating Circle.*

PROP. I.

If F *be any point in the curve, then will* CF : *radius* CA : : *arch* AB : *circumference* ABDA. 30.

For ſince the motion of the radius, and the moving point are both uniform, and the radius moves over AB, while the point moves over CF; alſo the radius moves over ABDA, whilſt the point moves over CA. Therefore by the laws of uniform motion, CF : CA : : arch AB : arch ABDA.

Cor. 1. *If the angles about* C *be taken in arithmetic progreſſion, the diſtances of the moving point from* C *will alſo be in arithmetic progreſſion.*

Cor.

Fig. 30. Cor. 2. *The angles about the center, are as the diſtances of the moving point from the center.*

Cor. 3. *The arches of the circle* ABD, *are as the diſtances of the moving point from the center.*

Cor. 4. *If radius* CA $= r$, *circumference* ABDA $= c$, *any arch* ABD $= z$, *and its ordinate* CP $= y$; *then the equation of the curve is,* $y = \frac{rz}{c}$.

PROP. II.

31. *To any point* P *of the curve, draw* CPD *from the center, and* CT *perp. to it; and let the radius* CA : *arch* ABD : : CP : CT. *Draw* PT, *and it will touch the curve in* P.

Put c = circumference ABDA, r = radius CA, z = arch ABD, y = CP. Draw Cm infinitely near CD, cutting the ſpiral in n, and draw Pr perp. to Cm. Then (Prop. 1. Cor. 3.) arch ABD : CP : : arch ABDm : Cn. And by diviſion, arch ABD : CP : : arch Dm : Cn — CP or nr; that is, $z : y :$ Dm : $nr = \frac{y}{z} \times$ Dm; and by ſimilar ſectors, CD (r) : CP (y) : : Dm : P$r = \frac{y}{r} \times$ Dm. But if PT be a tangent, the triangles nPr and PTC are ſimilar; therefore nr $\left(\frac{y}{z} \times \text{D}m\right)$: rP $\left(\frac{y}{r} \times \text{D}m\right)$: : PC (y) : CT $= \frac{zy}{r}$. Whence $r : z : : y :$ CT.

Cor. *If* CT *be perp. to* CP, *and the arch* POL *be deſcribed with the radius* CP. *And if* CT *be made equal to the arch* POL; *then* TP *will be a tangent at* P.

For by ſimilar ſectors, CD (r) : CP (y) : : arch ABD (z) : arch POL $= \frac{yz}{r} =$ CT by this Prop. Fig. 31.

PROP. III.

Let F *be any point in the curve; with radius* CF, *deſcribe the arch* FG; *then* $AC^2 : CF^2 :: $ *circumference* ABD : *arch* FG. 32.

For let CA $= r$, circumference $= c$, arch AB $= z$, arch GF $= v$, CF $= y$. Then (Prop. I.) circumference (c) : radius (r) : : arch AB (z) : CF (y); and by ſimilar ſectors, CF (y) : FG (v) : : CA (r) : arch AB (z). And multiplying, $cy : rv :: rz : zy :: r : y$. And taking the product of the extremes and means, $cyy = rrv$, and $rr : yy :: c : v$.

Cor. *Hence another equation of the curve is* $yy = \frac{rrv}{c}$.

PROP IV.

The ſpiral ſpace CPNC *is equal to* $\frac{1}{3}$ *the ſector* CNQL, *whoſe radius is* CN, *and arch* NQL. 33.

Draw CIS infinitely near CPQ, and Pr perp. to CS. Then CQ^2 or $CN^2 : CP^2 ::$ ſector CQS : ſector of the ſpiral CPI $= \frac{CP^2}{CN^2} \times$ CQS. Therefore the ſum of all the ſectors CPI = ſum of all the $\frac{CQS}{CN^2} \times CP^2$. But if the arch NQL be divided into an infinite number of equal parts, equal to SQ, and lines drawn from the center; all the ſectors formed thereby will be equal. And all the CP's will be in arithmetic progreſſion. Therefore (Arith. Infin. Prop. III.) the ſum of all the $CP^2 =$ $\frac{1}{3} CN^2$

Fig. 33. $\frac{1}{3}CN^2 \times$ by the number of them. And the ſum of all the $\frac{CQS}{CN^2} \times CP^2 = CQS \times \frac{1}{3}$ the number of them. Whence the ſum of all the CPI or the ſpiral ſpace CPNC $= \frac{1}{3}$ CQS × number of ſectors CQS $= \frac{1}{3}$ the whole ſector CLQN.

Cor. *The whole ſpiral ſpace* CPNA $= \frac{1}{3}$ *the generating circle* ABDA.

For then CN becomes equal to CA.

PROP. V.

34. *If the area of the firſt circle be* A, *the firſt, ſecond, third, fourth, &c. ſpiral ſpace, will be* $\frac{1}{3}$ A, $\frac{7}{3}$ A, $\frac{19}{3}$ A, $\frac{37}{3}$ A, &c.

It muſt be conſidered that in deſcribing the ſecond ſpiral CFGALMN, the firſt ſpiral ſpace is twice deſcribed, becauſe there are two revolutions. Alſo in deſcribing the third ſpiral, the firſt ſpiral ſpace is thrice deſcribed, and the ſecond twice. Likewiſe in deſcribing the fourth ſpiral, the firſt ſpiral ſpace is deſcribed four times; the ſecond thrice, the third twice; and ſo on. Likewiſe in deſcribing the ſecond, third, fourth, ſpiral; the correſponding circle is twice, thrice, or four times deſcribed, reſpectively; by reaſon there are ſo many revolutions.

Fig. 34.

Therefore let them be denoted as follows.

orders	ſimple ſpiral ſpaces	circles' areas	ratio of the circles
1	a	A	A = A
2	b	B	B = 4A
3	c	C	C = 9A
4	d	D	D = 16A
5	e	E	E = 25A
&c.			

Then by Prop. IV. and what is now mentioned, we ſhall have

$$a = \tfrac{1}{3}\,\mathrm{A}$$
$$a + b = \tfrac{1}{3} \times 2\mathrm{B}.$$
$$a + b + c = \tfrac{1}{3} \times 3\mathrm{C}.$$
$$a + b + c + d = \tfrac{1}{3} \times 4\mathrm{D}.$$
$$a + b + c + d + e = \tfrac{1}{3} \times 5\mathrm{E}.$$

&c.

Then ſubtracting each term from the following one, we ſhall have, the ſimple ſpiral ſpaces, viz.

$$a = \frac{1}{3}\,\mathrm{A}$$
$$b = \frac{2}{3}\mathrm{B} - \frac{1}{3}\mathrm{A} = \frac{8}{3}\,\mathrm{A} - \frac{1}{3}\mathrm{A} = \frac{7}{3}\,\mathrm{A}.$$
$$c = \frac{3}{3}\mathrm{C} - \frac{2}{3}\mathrm{B} = \frac{27}{3}\,\mathrm{A} - \frac{8}{3}\,\mathrm{A} = \frac{19}{3}\,\mathrm{A}.$$
$$d = \frac{4}{3}\mathrm{D} - \frac{3}{3}\mathrm{C} = \frac{64}{3}\mathrm{A} - \frac{27}{3}\mathrm{A} = \frac{37}{3}\,\mathrm{A}.$$
$$e = \frac{5}{3}\mathrm{E} - \frac{4}{3}\mathrm{D} = \frac{125}{3}\,\mathrm{A} - \frac{64}{3}\mathrm{A} = \frac{61}{3}\,\mathrm{A}.$$

Cor. 1. *If a, b, c, d, &c. be the firſt, ſecond, third, fourth, &c. ſpiral ſpace, and* A, B, C, D, *&c. the correſpondent circles. Then will a, b, c, d, &c. be reſpectively equal to* $\frac{1}{3}$ A, $\frac{7}{12}$ B, $\frac{19}{27}$ C, $\frac{37}{48}$, D, *&c.*

Fig. 34. Cor. 2. *The area of the nth ſpiral, will be* $\frac{1}{3}A \times : 3n \times \overline{n-1} + 1 :$

Take the ſeveral differences as follows,

		1 *dif.*	2 *dif.*	3 *dif.*
1	$\frac{1}{3}$ A	2 A	2 A	0
2	$\frac{7}{3}$ A	4 A	2 A	0
3	$\frac{19}{3}$ A	6 A	2 A	
4	$\frac{37}{3}$ A	8 A		
5	$\frac{61}{3}$ A			

Then by Prop. II. of the differential method, the nth ſpiral ſpace is $= \frac{1}{3}A + \frac{n-1}{1} \times 2A + \frac{n-1}{1} \times \frac{n-2}{2} \times 2A = \frac{1}{3}A : \times 1 + 6 \times \frac{n-1}{1} + 6 \times \frac{n-1}{1} \times \frac{n-2}{2} = \frac{1}{3}A \times : 1 + 6 \times \frac{n-1}{1} \times \overline{1 + \frac{n-2}{2}} = \frac{1}{3}A \times : 1 + 6 \times \frac{n-1}{1} \times \frac{2+n-2}{2} = \frac{1}{3}A \times : 1 + 3n \times \overline{n-1} = \frac{1}{3}A + nA \times \overline{n-1}.$

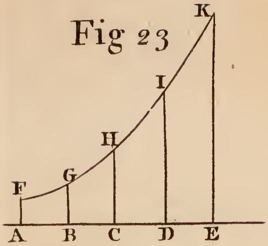
Cor. 3. *If* P *be the nth circle. Then the nth ſpiral ſpace is equal to* $\frac{n \times \overline{n-1} + \frac{1}{3}}{nn}$ P.

For $nnA = P$, and $A = \frac{P}{nn}$.

Cor. 4. *The ſpiral ſpaces contained between every two ſucceeding circles, are* 2A, 4A, 6A, 8A, &c.

For theſe are the firſt differences of the whole ſpaces.

Fig 23



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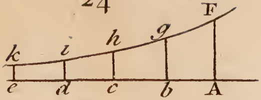

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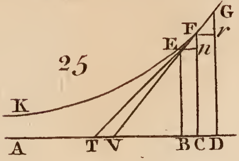

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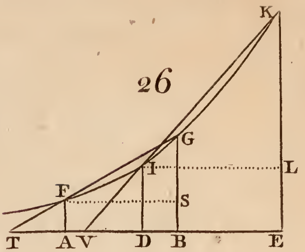

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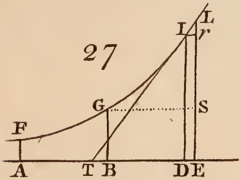

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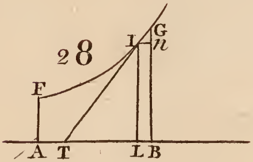

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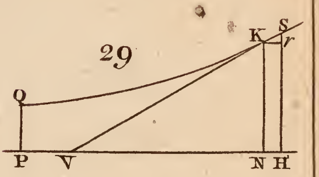

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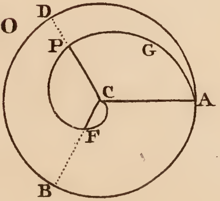

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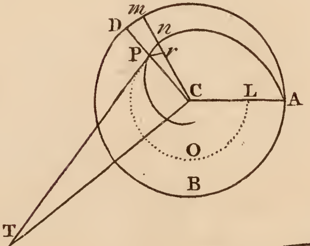

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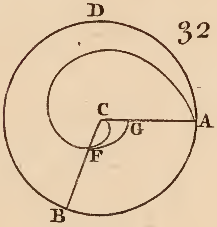

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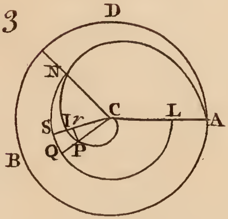

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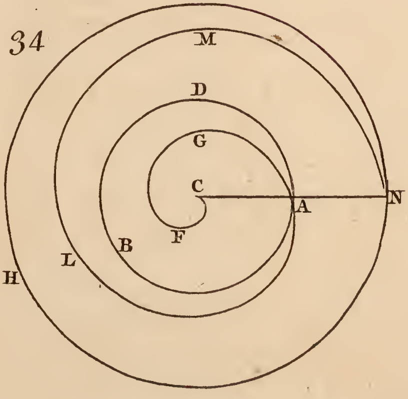


Pl. III. *pa.* 30.

S E C T. VII.

Of the LOGARITHMIC SPIRAL.

DEFINITIONS.

DEF. I.

IF the curve FGDA be drawn about the center C, ſo that it may every where cut the radii CF, CD, CG, CA, &c. in one and the ſame angle; that curve is called the *Logarithmic* or *Proportional Spiral.* Fig. 35.

DEF. II.

The lines CF, CD, CG, &c. are called *Ordinates.*

PROP. I.

If any number of ordinates CF, CK, CD, CA, *&c. be drawn, making equal angles at the center* C. *Then theſe ordinates ſhall be in continual proportion.* 36.

Suppoſe an infinite number of ordinate CF, CG, CH, CI, &c. to be drawn, making equal angles at the center. Then the triangles CFG, CGH, CHI, CIK, &c. are ſimilar; for the angles at G, H, I, K, are equal (by def. I.), and the angles at the center are all equal by conſtruction. Whence CF : CG : : CG : CH : : CH : CI : : CI : CK, &c. continued as far as CA. But ſince the angles FCK, KCD, DCA are equal; as many continual proportionals may be interpoſed between KC and DC, and alſo between DC and CA, as there are between FC and KC. Therefore CF, CK, CD, CA are equidiſtant terms. But in a ſeries of terms in continual proportion; the equidiſtant terms are alſo in continual proportion (Propor. Prop. 21.) Therefore CF : CK : : CK : CD : : CD : CA.

Fig. 36. Cor. 1. *If the angles at the center be in arithmetic progreſſion, the ordinates will be in geometrical progreſſion.*

Cor. 2. *The ſpiral makes an infinite number of revolutions, before it comes to the center* C.

For theſe proportional ordinates are continued ad infinitum downwards.

Cor. 3. *Let any radius* CA *cut the ſpiral in its ſeveral revolutions; then the diſtances of the ſeveral points of interſection from the center, will be in geometrical progreſſion, and that in the ratio of* CA *to* CB.

This appears by Cor. 1. for each revolution makes four right angles.

PROP. II.

37. *If* CD, CE, CF, CA, *&c. be in geometrical progreſſion, and* CD = 1; *then will the angles* DCE, DCF, DCA, *&c. be the logarithms of* CE, CF, CA, *&c.*

For ſince CD, CE, CF, CA are in geometrical progreſſion; the angles DCE, ECF, FCA, are equal, and the angles DCE, DCF, DCA, in arithmetic progreſſion. But it is the property of logarithms to be in arithmetic progreſſion, when the numbers anſwering are in geometrical progreſſion. Therefore theſe angles are the logarithms of the ordinates.

38. Cor. *Let* CF, CG, CA, *be any ordinates; then angle* FCG : *angle* FCA : : *log.* : $\frac{CG}{CF}$: *log.* : $\frac{CA}{CF}$.

For let CD = 1. Then angle DCA = log. CA,
and angle DCG = log. CG,
alſo angle DCF = log. CF,

and by ſubtraction, Fig. 38.

$\angle$ DCG — DCF = log. : CG — log. : CF,

that is, angle FCG = log. : $\frac{CG}{CF}$.

Alſo $\angle$ DCA — DCF = log. : CA — log. : CF,

that is, angle FCA = log. : $\frac{CA}{CF}$.

Therefore in any ſyſtem of logarithms it will be, angle FCG : angle FCA : : log. : $\frac{CG}{CF}$: log. : $\frac{CA}{CF}$.

PROP. III.

If CM *is any ordinate, and* CT *perp. to it; and if the angle* CMT *be made equal to the angle of the ſpiral. Then* MT *is a tangent at* M. 39.

For the tangent MT, and the infinitely ſmall part of the curve at M, coincide.

Cor. TM *is to* CM *every where in the ſame given ratio.*

PROP. IV.

If MT *be a tangent at* M, CT *perp. to the ordinate* CM; *then will the length of the whole ſpiral* MDGC *be equal to the tangent* TM. 39.

Take the point *n* infinitely near M, and draw *nr* perp. to CM. Then in the right angled triangle *n*M*r*, all the angles are given, therefore M*r* is to M*n* in a given ratio, ſuppoſe as *r* to *s*, then M*n* $= \frac{s}{r} \times$ M*r*. And all the M*n* $= \frac{s}{r} \times$ all the M*r*, that is, the length of the curve $= \frac{s}{r} \times$ CM. But the triangles *n*M*r* and TMC are ſimilar, whence

TM :

Fig. TM : CM : : nM : Mr : : s : r; therefore TM $=$
39. $\frac{s}{r} \times$ CM $=$ length of the whole curve.

PROP. V.

40. *Let m*D *be a tangent at m, and* CD *perp. to m*D; CM, Cm, *ordinates. Then m*D : mC : : CM — Cm : Mm *the length of the ſpiral contained between the ordinates.*

Draw CT, Ct perp. to CM, Cm, produce mD to t, and draw the tangent MT. Then from the laſt Prop. TM $= \frac{s}{r}$ CM, and $tm = \frac{s}{r}$ Cm, and TM — $tm = \frac{s}{r} \times \overline{\text{CM} - \text{C}m}$. But (Prop. IV.) TM $=$ curve MGC, and $tm =$ curve MGC; therefore TM — $tm =$ the part of the curve Mm; therefore M$m = \frac{s}{r} \times \overline{\text{CM} - \text{C}m}$, and CM — C$m$: Mm : : r : s : : Cm : tm : : (ſimilar triangles) mD : mC.

Cor. *Hence the parts of the curve are as the differences of the ordinates including them.*

PROP. VI.

41. *If* CM *be an ordinate,* MT *a tangent,* CT *perp. to* CM. *Then the area of the ſpiral* MDC, *deſcribed by a radius making an infinite number of revolutions, is half the triangle* TCM.

Draw the ordinate nC infinitely near MC, and nr perp. to CM. Then ſince all the angles of the triangle nMr are given, Mr is to nr in a given ratio, ſuppoſe as r to q; then $nr = \frac{q}{r} \times$ Mr. And

the

Fig. 41. the area of the triangle MCn $= \frac{MC \times nr}{2} = \frac{q}{2r} \times$ MC × Mr. Divide MC into an infinite number of equal parts equal to Mr. Then the ſum of all the MC, nC, &c. to C, is equal to half ſo many times the greateſt MC; that is, (as the number of them is denoted by MC,) equal to $\frac{MC}{2} \times$ MC or $\frac{MC^2}{2}$. And the ſum of all the $\frac{q}{2r} \times MC \times Mr = \frac{q}{2r} \times \frac{MC^2}{2}$ or $\frac{q}{4r} \times MC^2$. Therefore the ſum of all the triangles MCn, or the whole area of the ſpiral ſpace $= \frac{q}{4r} \times MC^2$. But by the ſimilar triangles Mnr, MTC; MC : TC : : Mr : nr : : $r : q$, therefore TC × r = MC × q, and $\frac{MC \times q}{r}$ = TC, and $\frac{q \times MC}{4r} = \frac{1}{4}$ TC. Whence the whole area of the ſpiral ſpace MDC $= \frac{1}{4}$ TC × MC $= \frac{1}{2}$ triangle TCM.

Cor. 1. *The ſpiral ſpace* MDFGM, *comprehended between the part of the ordinate* GM, *and the curve* MDFG; *is equal to* $\frac{TC}{4MC} \times \overline{MC^2 - GC^2}$.

For the ſpiral ſpace MDGC $= \frac{q}{4r} \times MC^2$, and the ſpace GC $= \frac{q}{4r} \times GC^2$, therefore the difference, viz. MDFG $= \frac{q}{4r} \times \overline{MC^2 - GC^2} = \frac{TC}{4MC} \times \overline{MC^2 - GC^2}$.

40. Cor. 2. *If* MC, mC *be two ordinates*; MT, mt, *two tangents*; CT, ct, *perp. to* CM, Cm. *Then the*

Fig. 40. *area* MCm, *comprehended between theſe ordinates, and the ſpiral; is equal to half the difference of the triangles* TCM *and* tCm; *or equal to* $\frac{TC}{4MC} \times \overline{CM^2} - \overline{Cm^2}$.

This follows from the Prop. and Cor. 1.

SCHOLIUM.

The area mentioned in this Prop. is that made by infinite revolutions of the radius; where every ſucceeding revolution covers a part of the former. And ſo the ſame parts of the area are deſcribed over and over. And therefore the area mentioned in Cor. 1. is the true area of the whole ſpiral; and is that contained between the curve and the part of the ordinate intercepted by the curve.

PROP. VII.

42. *In the ſpiral* CAB, *if the ordinate* CA = 1, *and the circle* ADE *be deſcribed with the radius* CA; *and any other ordinate* CB *be drawn. Then arch* AD = *log. of* CB.

For ſince AC = 1, therefore (Prop. II.) the angle ACD is the logarithm of the ordinate CB. But the angles at C, being proportional to the arches of the circle; therefore the arch AD will alſo be the logarithm of CB, according to ſome ſpecies or other of logarithms.

Cor. 1. *From the point* A *where it cuts the circle, draw the tangent* AT, *and* CT *perp. to* CA. *Then if the ſubtangent* CT *be* .43429448, *then* AD *is Brig's logarithm of the number* CB.

For by the nature of logarithms (demonſtrated Cor. 2. Prop. VIII. Sect. V.), the exceſs of a number (extremely near 1) above 1 : is to its logarithm : :

1 :

1 : ſubtangent. Therefore draw the ordinate C*rn* infinitely near CA. Then the triangles *n*A*r*, ATC, are ſimilar; whence *nr* : A*r* : : AC : CT. But *nr* is the exceſs of C*n* above 1, and A*r* is its logarithm; and AC is 1, and CT the ſubtangent = .43429448; and (Prop. VII. Cor. Sect. V.) this is the ſubtangent to Brig's logarithms. Therefore A*r* is Brig's logarithm of C*n*, and conſequently AD is Brig's logarithm of CB. Fig. 42.

Cor. 2. *If the ſubtangent* CT *be* 1; *then* AD *is Napier's logarithm of* CB. *In other caſes* AD *will belong to ſome other ſpecies of logarithms.*

For (Cor. Prop. VII. Sect. V.) 1 is the ſubtangent of Napier's logarithms.

SECT.

SECT. VIII.

Of the HYPERBOLIC SPIRAL.

DEFINITIONS.

DEF. I.

Fig. 43. DRAW any line CG; and from C as a center, deſcribe as many arches of circles as you pleaſe from CG, as VE, GF, *gf*, &c. all equal to one another. Then the curve paſſing thro' all the points which terminate theſe arches, as FE*fd*C is called the *Hyperbolic* or *Reciprocal Spiral.*

DEF. II.

Any right line drawn from the center to the curve, as CF, is called an *Ordinate.*

DEF. III.

The line CG is called the *Axis.*

PROP. I.

44. *If* CA *be perp. to the axis* GC, *and equal to any of the arches* VE, GF, *&c. and* AB *be parallel to* CG. *Then* AB *is an aſſymptote to the curve.*

Thro' F draw DB parallel to CA, then ſince DB = CA = GF, therefore BF = GF — DF = the difference between the arch and the ſine, but when the radius CG is encreaſed infinitely, that difference FB is infinitely diminiſhed, and becomes leſs than any given line. Therefore AB is an aſſymptote.

Cor. *The diſtance of the aſſymptote from the axis, is equal to any arch* GF, *between the axis and curve.*

PROP.

Fig.

PROP. II.

Thro' any point F *of the curve, draw the arch* GF *and the ordinate* CF; *and draw any other ordinate* CH, *cutting the arch* GF *in* L. *Then will* $\text{CF} \times$ *arch* $\text{GF} = \text{CH} \times$ *arch* GL. 45.

Thro' H deſcribe the arch HI. Then the ſectors CHI, CLG, are ſimilar; whence CH or CI : CG : : arch IH or GF : arch GL. Whence $\text{CH} \times$ arch $\text{GL} = \text{CG} \times$ arch GF.

Cor. 1. $\text{CI} \times$ *arch* $\text{IH} = \text{CG} \times$ *arch* IN.
For CI : CG : : IN : GF or IH.

Cor. 2. If radius $\text{CG} = r$, CA or arch $\text{GF} = a$, ordinate $\text{CH} = y$, arch $\text{GL} = z$. Then the equation of the curve is $ar = zy$.

PROP. III.

If HT *be a tangent at any point* H, *and* CT *be drawn perp. to* CH. *Then the ſubtangent* CT *is equal to the arch* IH. 46.

With any radius CG deſcribe the arch GFL, cutting the ordinate CH in L, and draw C*hl* infinitely near CHL. Then (Prop. II.) $\text{CG} \times \text{GF} = \text{CH} \times \text{GL} = \text{C}r \times \text{G}l = \overline{\text{CH} - rh} \times \overline{\text{GL} + \text{L}l}$ = (becauſe *rh*, L*l*, are infinitely ſmall) $\text{CH} \times \text{GL} + \text{CH} \times \text{L}l - \text{GL} \times rh$. Whence $\text{CH} \times \text{L}l = \text{GL} \times rh$. But the ſectors CH*h*, CL*l* are ſimilar, whence CH : H*h* : : CL : L*l*, and $\text{CL} \times \text{H}h = \text{CH} \times \text{L}l = \text{GL} \times rh$. Alſo the triangles CHT and *hr*H are ſimilar; therefore *rh* : *h*H : : HC : CT, and $\text{CT} \times rh = \text{HC} \times \text{H}h$; and $\text{CL} \times \text{CT} \times rh = \text{CH} \times \text{CL} \times \text{H}h = \text{CH} \times \text{GL} \times rh =$ (Prop. II.) $\text{CL} \times \text{GF} \times rh$. Therefore $\text{CT} = \text{GF}$ = (Defin. 1.) IH.

Cor.

Fig. 46. Cor. 1. *The ſubtangent* CT = CA *the diſtance of the aſſymptote, and therefore is always the ſame.*

Cor. 2. *The angle* CHT *continually increaſes, in approaching to the center.*

For CT is given, and CH continually decreaſes.

Cor. *Hence the curve makes an infinite number of revolutions about the center.*

For in the logarithmic ſpiral, where the angle at H is perpetually the ſame, the curve makes an infinite number of revolutions. And much more does it ſo in this, where the angle at H continually increaſes, and makes it keep without the logarithmic ſpiral.

PROP. IV.

45. *If* CF, CH, *be any two ordinates, theſe ordinates are reciprocally as the angles they make with the axis* CG. CF : CH : : $<$ GCH : $<$ GCF.

For (Prop. II.) CF × GF = CH × GL. Therefore CF : CH : : arch GL : arch GF : $<$ GCL : $<$ GCF.

PROP. V

25. 47. *The length of the ſpiral between any two ordinates, is equal to the length of the logarithmic curve between two ordinates of equal lengths with the former; when the ſubtangents of both curves are the ſame.*

Let HDC be the ſpiral, CEK the logarithmic curve; let the ordinate CH, C*n* and DG, CF be infinitely near, and CH = DG, and C*n* = CF, in the two curves. With the radius C*n* deſcribe the arch *nm*; alſo draw F*r* parallel to AD. Then ſince the ſubtangent VD = CT, alſo DG = CH. Therefore the right angled triangles CTH and DVG are equal and ſimilar, whence the angle CHT = DGV. And ſince C*n* = CF, therefore CH — C*n* or *m*H = DG — CF or *r*G, and ſince $< \mathrm{H} = < \mathrm{G}$, therefore the triangles *m*H*n* and FG*r* are equal and ſimilar; and H*n* = GF; that is, the curve H*n* contained between the ordinates CH, C*n*, is equal to the curve GF contained between the ordinates DG, CF equal to the former; and ſo it will always be. Therefore the ſum of all the H*n*, or the ſpiral contained between any two ordinates, is equal to the ſum of all the GF, or the log. curve contained between two ordinates equal to the former. Fig. 47.

Cor. *The ſpiral making an infinite number of revolutions, is alſo infinite in length.*

For the logarithmic curve, and its aſſymptote is infinite in length; and conſequently the ſpiral, which is equal to it, is the ſame.

PROP. VI.

The area of the whole ſpiral FLDC, *making infinite revolutions, is equal to* $\frac{\mathrm{CF} \times \mathrm{CT}}{2}$, *half the rectangle of the ordinate aod ſubtangent.* 48.

Draw the ordinate C*r* infinitely near CF, and *rn* perp. to CF. And by ſimilar triangles F*n* : *nr* : : CF : CT the ſubtangent. Whence $\mathrm{CT} \times \mathrm{F}n = \mathrm{CF} \times nr$; but the triangle $\mathrm{FC}r = \frac{\mathrm{CF} \times nr}{2}$; therefore the triangle $\mathrm{FC}r = \frac{\mathrm{CT} \times \mathrm{F}n}{2}$. And the ſum of all the triangles FC*r* deſcribed in all the revolutions $= \frac{1}{2}\,\mathrm{CT} \times$ all the F*n*. That is, the whole area of the ſpiral ſpace $= \frac{1}{2}\,\mathrm{CT} \times \mathrm{FC}$.

Cor.

Fig. 48. Cor. 1. *The ſpiral ſpace* FLNDF *contained between the curve, and part of the ordinate* DF, *is equal to* $\frac{1}{2} CT \times DF$.

For the whole ſpiral ſpace FLNDC $= \frac{1}{2} CT \times CF$, and the ſpace DSC, which is covered by the other, is $= \frac{1}{2} CT \times CD$. And their difference $= \frac{1}{2} CT \times \overline{CF - CD} = \frac{1}{2} CT \times DF$.

Cor. 2. *Any part of the ſpiral ſpace contained between two ordinates* CF, CL, *is equal to* $\frac{1}{2} CT \times \overline{CF - CL}$.

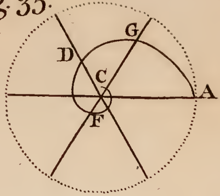
For $\frac{1}{2} CT \times CF =$ ſpace FLNDC, and $\frac{1}{2} CT \times CL =$ LNDC. And their difference $= \frac{1}{2} CT \times \overline{CF - CL}$.

Cor. 3. *The ſpiral ſpace contained between any two ordinates, is half the ſpace in the logarithmic curve contained between two ordinates, equal to the former; when the ſubtangents of both curves are the ſame.*

For (Prop. IV. Sect. V.) the logarithmic ſpace is equal to the rectangle of the ſubtangent and difference of the ordinates, and in the ſpiral it is equal to half that rectangle.

I might here have given more properties of all theſe curves, and likewiſe have ſhewn the nature and properties of ſome other ſorts of curves, as the *Catenary*, the *Epicycloid*, &c. But as this cannot eaſily be done without having recourſe to the method of *Fluxions*; I forbear purſuing them any further, as the making uſe of any other method, would be carrying the reader too far about.

Fig. 35.



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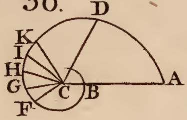

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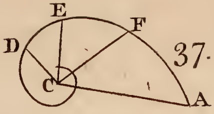

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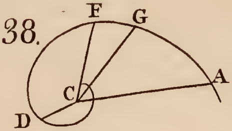

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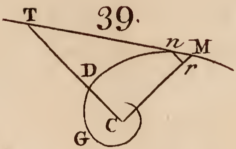

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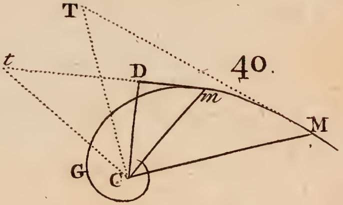

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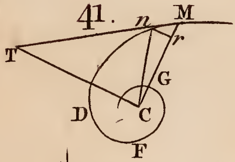

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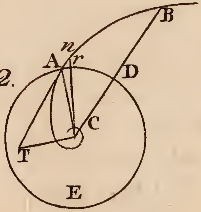

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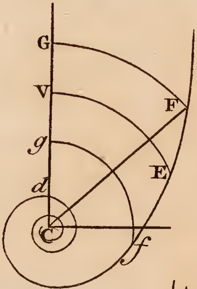

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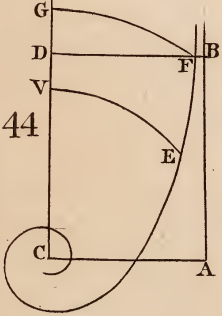

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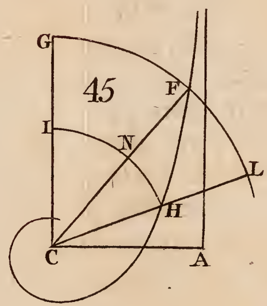

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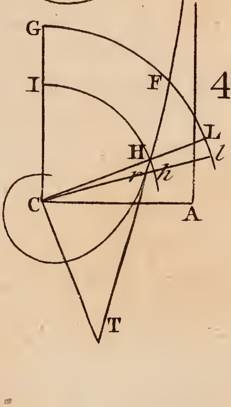

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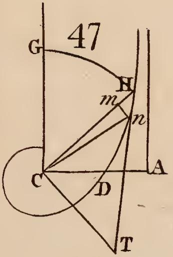

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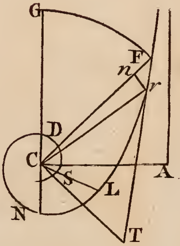


Pl. IV *pa*. 42.

BOOK II.

Of Curve Lines *in General; and their Affections.*

CURVE lines may be conceived to be generated theſe three ways. Firſt, they may be made by the ſection of a ſolid and a plane. Secondly, they may be defined by an equation, expreſſing their nature, or ſome principal property thereof. Thirdly, they may be conceived to be deſcribed *in plano* by local motion.

1. Firſt, thoſe who conſider curve lines as generated in the ſurface of a ſolid by a plane cutting it; find it proper and neceſſary to conſider the properties of that ſolid; and to demonſtrate from thence what muſt be the nature of a figure ariſing from ſuch a ſection. Accordingly the ancient geometricians conſidered the conic ſections, as made by a plane cutting a cone.

2. In the ſecond caſe, equations, which expreſs the nature of curve lines, are not properly definitions of theſe curves, but certain arithmetical computations, grounded on ſome property of theſe curves, and which (duly managed) may lead us to the knowledge of theſe properties. For theſe equations, being analytical expreſſions, will (like the ſcience itſelf) lead us in a retrograde order from the equations themſelves, to the properties on which they are founded. For it is well known in analytics, that by conſidering the things ſought, as if they were known; at laſt, by equations expreſſing their conditions and relations, we arrive at the real knowledge of them. Juſt thus in geometry, by

denoting the conditions of a curve by an equation expreſſing its nature, we at laſt arrive at ſome property of that curve, upon which that equation is founded. Therefore it is plain, that equations are not true definitions of a curve, but only artificial expreſſions, by the help whereof, and the given conditions, we are able to trace the firſt, moſt ſimple, and defining property of ſuch curve. An equation then is nothing elſe but a deſcription, by which a figure is rendered more eaſy to the conception. When the nature of a curve is expreſſed by an equation, we muſt find out by what motion of points, interſection of lines, or by what deſcription or conſtruction, a curve is generated, which ſhall have the conditions expreſſed in that equation. An equation then pre-ſuppoſes a curve deſcribed according to ſome law or rule, upon which that equation is founded.

3. Thirdly, The deſcription of a curve in plano, is the moſt true and natural definition of a curve, and the proper original thereof, and from which all other properties may be determined. Now this conſtruction may be effected various ways, as by the interſection of lines, the motion of points, ſimple or compound after ſeveral manners, which will generate ſeveral ſorts of curves. Thus a circle is deſcribed by a line revolving round a fixed center; an ellipſis is deſcribed by a flexible line moving round two fixed centers; a cycloid is deſcribed by a point in the circumference of a circle, whilſt it turns round along a right line; and ſo of others.

The ancients admitted only right lines, the circle, and the conic ſections into geometry, excluding all other curves as not geometrical. But modern geometers admit all curves into geometry, that can any way be expreſſed by equations. Curves are beſt divided into kinds by the dimenſions of the equations expreſſing their nature. By help of theſe quations, we gain a better idea of ſuch figures, nd more eaſily comprehend the nature of them.

DEFINITIONS.

DEF. I.

A Curve Line is that which is deſcribed by a Fig. noving point which continually changes its direc- 49. ion, as ABC; whilſt a *right line* always goes on in he ſame direction, as AD.

DEF. II.

If BM be any curve line, and any right line AD 50. eing drawn of an indefinite length, and having ts origin in A; and if any part AP be cut off, and rom P, the right line PM be drawn to the curve, naking a given angle with AP; and this be every vhere done; then the line AP is called the *Ab-ciſſa*, and PM the *Ordinate*, of that curve.

DEF. III.

A line of any *Order* is denominated from the igheſt power of the equuation, expreſſing the na-ure thereof; which powers are made of the ab-ciſſa, or the ordinate or both.

Thus if $x =$ abſciſſa; $y =$ ordinate; a, b, c, iven quantities; if $ax = by + cc$, this denotes a ine of the *firſt order*, or a right line. If $yy - ay = bx + xx$, this denotes a line of the *ſecond order*, r a curve of the firſt order. And ſo does this quation $ay + xy = bx$. And this $ayy = bxy - ccx$.

Again, $y^3 - 3axy = bxx$, denotes a line of the *hird order*, or a curve of the ſecond order. And o on to higher orders, according to the dimen-ions of x or y, or both of them in one term.

DEF. IV.

If x be the abſciſſa, and y the ordinate of any curve; then if the equation expreſſing the nature

Fig. 50. of the curve, be made up of x and y and given quantities, and x and y denote ſtreight lines, then that curve is called a *Geometrical* or *Algebraic Curve.*

DEF. V.

But if the nature of a curve be expreſſed by an equation, wherein one of the indetermined quantities x or y, repreſents a curve line, then the curve denoted by ſuch an equation, is called a *Tranſcendent* or *Mechanical Curve.*

DEF. VI.

Exponential Curves, are thoſe where the equations expreſſing the nature thereof conſiſt of terms whoſe indices are indetermined or variable quantities made of x or y. Thus $a^x = y$, is an exponential curve, as alſo $y^x + x^y = axy$.

DEF. VII.

51. Curves that turn round a fixed point or center, are called *Spirals* or *Radial Curves*, as ADE. And any line CE, drawn from the center C to the curve, is called an *Ordinate.*

DEF. VIII.

A tangent to a curve, is a line drawn thro' a point of the curve without entering into it; ſo that being produced, it leaves the curve on both ſides.

DEF. IX.

52. *The Subtangent* is the line PT, or the diſtance of the ordinate from the interſection of the tangent and abſciſſa:

51. And in ſpirals, it is the line CT, which is perp. to the ordinate CE, and meets the tangent at T.

DEF.

DEF. X. Fig. 52.

The *Subnormal* is the line PQ, which is the diſtance of the ordinate from that point of the abſciſſa, where a perpendicular to the curve (or tangent) interſects it.

And in the ſpiral, it is the line CQ, perp. to 51. CE, and cutting the line EQ (which is perp. to the curve) in Q.

DEF. XI.

The *Axis* of a figure is a line paſſing thro' the center, which is ſome remarkable point of the curve; if it biſſects the ordinates, it is called a *Diameter.*

DEF. XII.

An *Aſſymptote* is a right line which continually approaches the curve, but never touches it, till it be at an infinite diſtance.

SCHOLIUM.

All curves are more ſimple or compounded, according as their equations conſiſt of fewer or more terms. But the equations are the more comprehenſive the more terms they contain; and become univerſal when all the powers and products of the abſciſſa and ordinate, are expreſſed therein. To give ſome inſtances, let $x =$ abſciſſa; $y =$ ordinate; a, b, c, d, &c. known quantities, which may be either affirmative or negative. Then a line of the firſt order will be denoted thus, $y + ax + b = 0$. A line of the ſecond order thus, $y^2 + \overline{ax + b} \times y + cxx + dx + e = 0$. Lines of the third order thus, $y^3 + \overline{ax + b} \times yy + \overline{cx^2 + dx + e} \times y + fx^3 + gx^2 + hx + k = 0$. And ſo on; and theſe equations are univerſal, and take in all caſes of theſe ſeveral orders. And in theſe ſeveral equations, Fig. 51. tions, any of the quantities *a*, *b*, *c*, &c. may be wanting; and then the curve changes into a more ſimple one of the ſame order.

When theſe coefficients are multiplied out at length, the number of terms in the equations of the ſeveral orders, will be in the firſt 3; in the ſecond 6; in the third 10; in the fourth 15, &c. which are thus generated, the firſt $= 1 + 2 = 3$; the ſecond $= 1 + 2 + 3 = 6$; the third $= 1 + 2 + 3 + 4 = 10$; the fourth $= 1 + 2 + 3 + 4 + 5 = 15$, and ſo on.

SECT. I.

The generation of Curves, by which the length of any given Curve is determined; the uſe of tangents in determining their areas, ſurfaces or ſolidities.

PROP. I.

If AOM *be any given curve,* AP *its axis,* MT *a tangent at* M. *And* BC *be ſuch a curve, that* MDC *be drawn parallel to* AP; *it may be,* AD : *tangent* MT : : AN *a given line* : CD. Fig. 53.

Then the ſpace ABCD *will be equal to the rectangle of* AN *and the curve line* AOM.

Take the point *m* infinitely near M, and draw *mc* parallel to MC. Produce the ordinate PM to *s*, and draw *r*C parallel to *s*M. Then the triangles MPT, M*sm* are ſimilar, whence M*s* : M*m* : : MP or AD : MT : : (conſtruction) AN : CD. And therefore CD × M*s* = M*m* × AN; that is, the rectangle CD*dr* or CD*dc* = rectangle M*m* × AN. Hence the ſum of all the rectangles CD*dc* = ſum of all the rectangles M*m* × AN; that is, the area ABCD = rectangle AN × curve line AOM.

Cor. 1. *Hence if any tangent* MT *be known, the correſponding ordinate* CD *will be known.*

Cor. 2. *And if the area* ABCD *can be had, the length of the curve* AOM *will alſo be had.*

Fig.

PROP. II.

54. *Let* AM *be any curve,* AP *the axis,* PM *the ordinate perp. to it; and let the curve* KZ *be of ſuch a nature, that taking any point* M *in the curve* AM, *and drawing the line* MB *perp. to the curve in* M, *cutting the axis in* B, *and producing* MP *to* Z; *it may be always* PM : MB : : *arch* AM : PZ. *Then the area* AKZP *will be equal to half the ſquare of the arch* AM.

Take the point m infinitely near M, and draw mB. Alſo draw ms parallel to MZ, and mr perp. to it. Then the triangles Mmr, MPB are ſimilar, and mr : Mm : : MP : MB : : (conſtruction) arch AM : PZ. Then Mm × arch AM $=$ mr × PZ $=$ P$ps$Z. And the ſum of all the Mm × AM $=$ ſum of all the P$ps$Z $=$ area AKZP.

Divide AM into an infinite number of equal parts each equal to Mm $= 1$. Then all the arches beginning at A, conſtitute a ſeries of quantities in arithmetic progreſſion, whoſe number is AM, and the greateſt alſo AM. Therefore the ſum of all the AM (or AM × Mm) $= \frac{AM}{2} \times AM = \frac{AM^2}{2}$, by Prop. II. of the Arithm. of Infinites. Therefore $\frac{AM^2}{2} =$ area AKZP.

PROP.

Fig.

PROP. III.

Let AIB *be any curve,* AD *the axis,* BD *the ordinate perp. to* AD. *And let* KL, CE *be two curves, ſo related to this, that producing* BD *to* E *and* L, *and taking a given right line* KN, *it may be* $DL^2 = DE^2 + KN^2$, *and that the rectangle* $KN \times BD =$ *area* ADEC. *Then the rectangle* KN × *curve* AIB = *area* ADLK. 55.

Take the point *b* infinitely near B, and draw *bdel* parallel to BDEL; and draw B*m*, E*n*, L*r* parallel to AD. And put $AD = x$, $DE = y$, $DL = v$, curve $AIB = z$, $BD = w$, and $KN = a$, then by conſtruction $vv = yy + aa$; and $aw =$ ADEC; that is, $a \times BD = ADEC$, and $a \times bd = AdeC$, and their difference $a \times mb = DdeE = DE \times Dd = y \times Dd = y \times Bm$, and ſquaring, $a^2 \times \overline{mb}^2 = y^2 \times Bm^2$; that is, $\overline{vv - yy} \times mb^2 = y^2 \times Bm^2$; and tranſpoſing, $vv \times mb^2 = yy \times \overline{mb^2 + Bm^2} = yy \times Bb^2$, and extracting the root $v \times mb = y \times Bb$, and $av \times mb = ay \times Bb$, or (becauſe $a \times mb = y \times Bm$) $vy \times Bm = ay \times Bb$, or $v \times Bm = a \times Bb$; that is, $DL \times Dd = a \times Bb$, and the ſum of all the $DL \times Dd = a \times$ ſum of all the B*b*; that is, the area $ADLK = a \times$ curve AIB.

Cor. *Hence if the areas of the curves,* ADLK, ADEC *be known; the length of the curve* AIB *will be known.*

Fig.

PROP. IV.

56. *If* $aefg$ *be any ſort of ſpiral,* af *an ordinate,* ta *perp. to it,* tf *a tangent,* HI *a given line; and it be made as* $af : at :: \mathrm{HI} : ql$; *and* aq *be made equal to* af, *and* ql *laid from* q *to* l, *in a line perp. to* aq, *and if this be every where done, and the curve* mlk *drawn And if the curve* EFG *be of ſuch a nature, that the ordinate* $\mathrm{CF} = af$, *and* $\mathrm{BE} = ae$, *&c. and that* $\mathrm{BC} \times \mathrm{HI} =$ *area* $mpql$. *Then will the length of the curve* EF $=$ *length of the ſpiral* ef. *And the area* BEFC $=$ *twice the area* aef.

For ſuppoſe ag infinitely near af, and DG infinitely near CF. Draw FN parallel to BD. Then ſince $\mathrm{CF} = af$, and $\mathrm{DG} = ag$, their difference $\mathrm{NG} = ng$. Put $\mathrm{HI} = a$. Then ſince $pmkr = a \times \mathrm{BD}$, and $pmlq = a \times \mathrm{BC}$, therefore the difference $qlkr = a \times \mathrm{CD}$; that is, $ql \times qr$ or $ng \times ql = a \times \mathrm{CD}$. But the triangles aft, ngf are ſimilar, whence $ng : nf :: af : at ::$ (conſtruction) $a : ql$; therefore $a \times nf = ng \times gl = a \times \mathrm{CD}$, therefore $nf = \mathrm{CD} = \mathrm{NF}$. Whence the triangles nfg and NFG are equal and ſimilar, and $\mathrm{FG} = fg$, and all the FG $=$ all the fg; or the curve EFG equal to the curve efg. And ſince $af = \mathrm{CF}$, and $ag = \mathrm{DG}$; therefore the triangle $afg =$ half the trapezium CDGF. And the ſum of all the triangles afg $=$ half the ſum of all the trapezia CG; that is, the area $aef =$ half the area EBCF.

PROP. V.

Let AMC *be any curve*; AB *the axis*; PM, CB, *ordinates*, MT *a tangent*. *Let the parallelogram* PMDB *be inſcribed in the curve*. *Then will* PMDB *be the greateſt poſſible inſcribed parallelogram*, *when the length* PB *is equal to the ſubtangent* PT. 57.

Take the point *m* infinitely near M, and draw *mp*, *md* parallel to MP, MD. Then we have another parallelogram *mp*B*d*, inſcribed in the curve. Every quantity is at its greateſt, when by varying the terms of it, the increaſe is equal to the decreaſe. The triangles TPM and M*rm* are ſimilar, whence TP : PM : : M*r* : *rm*. Therefore PM × M*r* = TP × *rm*. Therefore if PB or MD be put equal to TP, we ſhall have PM × M*r* = MD × *rm* = *r*D × *rm*. That is, the parallelogram M*p* = par. *m*D. But when the parallelogram MPBD is changed into the parallelogram *mp*B*d*. The former is decreaſed by the ſpace M*p*, and increaſed by the ſpace *m*D = M*p*. Therefore, ſince the infinitely ſmall increaſe, is equal to the infinitely ſmall decreaſe; the parallelogram PMDB is a maximum.

Cor. *By a like reaſoning*, *the greateſt cylinder* PMDB, *that can be inſcribed in the ſolid*, *generated by the curve* AMC, *revolving about its axis* AB; *is when the length* PB *is equal to half the ſubtangent* PT.

For let $c = 3.1416$, then ſince PM × M*r* = TP × *rm*; let PB = $\frac{1}{2}$ PT, then TP = 2PB, and PM × M*r* = 2PB × *rm*; multiply both by $c \times$ PM, then $c \times \text{PM}^2 \times \text{M}r = 2c \times \text{PM} \times rm \times \text{PB}$. But $c \times \text{PM}^2 \times \text{M}r$ = ſolid deſcribed by PM*mp*; and $2c \times \text{PM} \times rm \times \text{PB}$ = ſolid deſcribed by DM*md*.

Therefore

Fig. 57. Therefore, the increaſe of the inſcribed cylinder being equal to the decreaſe, the cylinder muſt be a maximum.

PROP. VI.

58. *In any curve, if the points* M, *m, be taken infinitely near together; and the ordinates* MP, *mp, drawn; and alſo* M*r*, *mn, parallel to the axis* AP. *Then either of the parallelograms* MP*pr*, *or n*P*pm, may be taken for the ſpace* MP*pm.*

59. *And in ſpirals, if* M, *m be infinitely near, and* AM, A*m drawn; and the ſmall arches* M*r*, *mn, deſcribed from the center* A, *which will be perp. to* AM, A*m. Then either of the triangles* AM*r*, A*mn, may be taken for the ſpace* AM*m.*

58. For ſince the hypothenuſe M*m* is infinitely ſmall, the ſides M*n*, *nm*, or M*r*, *rm*, will much more be infinitely ſmall. But the parallelogram MP*pr* is to the parallelogram M*rmn*, as PM to M*n*; and as M*n* is infinitely leſs than MP; the parallelogram M*rmn* is infinitely leſs than MP*pr*; and much more is the triangle M*nm*, or M*rm* infinitely leſs than the ſpace MP*pm*; and therefore vaniſhes in reſpect of it, conſequently the par. MP*pr* = ſpace MP*pm* = par. *n*P*pm*.

59. In like manner in the ſpiral, the triangle M*mn* or M*rm* is infinitely leſs than AM*r* or A*mn*. And therefore AM*r* or A*mn*, may be taken for AM*m*, the difference being infinitely ſmall, or nothing, in reſpect of any of them.

58. Cor. 1. *If a curve* APM *be divided into an infinite number of parts by the ordinates* MP, *mp, &c. infinitely near one another; either the ſum of the circumſcribing parallelograms, as* P*pmn*; *or the ſum of the inſcribed parallelograms, as* P*pr*M; *may be taken for the area of the curve.*

And in ſpirals, either the ſum of the circumſcribing triangles Amn, *or of the inſcribed triangles* AMr, Fig. 59.
may be taken for the area.

For when the ordinates are parallel, the ſum of the circumſcribing and inſcribed parallelograms, differ from one another only by one of the parallelograms *nPpm*, which is infinitely leſs than the area APM, as P*p* is infinitely leſs than AP. And much more will the difference between the curve and either ſum, be infinitely ſmall.

For the ſame reaſon in ſpirals, the ſum of the circumſcribed, and the ſum of the inſcribed triangles, differ from the curve by leſs than the infinitely ſmall triangle AM*m*.

Cor. 2. *The infinitely ſmall part* M*m of the curve* AM *may be looked on as a right line.*

Becauſe its deviation from a right line muſt be infinitely ſmall in that ſhort ſpace.

PROP. VII.

Let AM *be any curve,* AP *the abſciſſa,* PM *an* 60.
ordinate, AT *a tangent at the vertex. And let* AN *be ſuch a curve, that drawing the tangent* MT *to any point* M, *to cut* AT *in* T, *and producing* MP, *make* PN = AT. *And this be every where done. Then drawing* AM, *the area of the curve* ANP *is twice the ſegment* ABMA.

Take *m* infinitely near M, and draw *mpn* parallel to MPN; alſo draw AQ*q* parallel to MT, then QM = AT = (by conſtruction) PN. Therefore the parallelogram QM*mq* = trapezium PN*np* (Prop. VI.). But the triangle AM*m* (contained between the ſame parallels AQ*q*, TM*m*) = half QM*mq* = half PN*np*. Therefore the ſum of all the triangles AM*m* = $\frac{1}{2}$ ſum of all the trapezia PN*np*; that is, the ſegment ABMA = half the area ANP.

Fig. 64. PZ. Then Mr × PZ = a × mr; that is, the trapezium P$pz$Z = a × mr. And the ſum of all the P$pz$Z or the area AKZP = a × ſum of all the mr = a × PM.

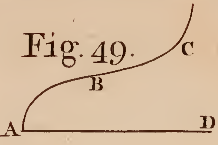
PROP. XII.

65. *Let* AM *be any curve,* AP *the abſciſſa,* PM *an ordinate,* MT *a tangent. And let the curves* BD, *and* GF *be ſo related, that drawing* CN *parallel to* PM, *and producing* MP *to* D, *and drawing* MNF *parallel to* AP; *it may always be,* PT : PM : : NF : PD. *Then the area* APDB *is equal to the area* CGFN.

Take m infinitely near M, and draw mpd parallel to MP, and mnf parallel to AP. Then the triangles Mmr, TMP, are ſimilar, whence Mr : mr : : TP : PM : : NF : PD. Therefore Mr × PD = mr × NF; that is, Pp × PD = Nn × NF, or which is the ſame thing, the trapezium P$pd$D = trapezium NFfn. And as this is ſo every where; the ſum of all the trapeziums P$pd$D = ſum of all the trapeziums NFfn, or the area ABDP = area CGFN.

Cor. *Hence if any two of the curves* AMP, ABDP *and* CGFN, *be given; the third may be found.*

Fig. 49.



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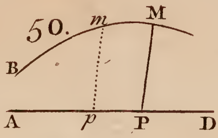

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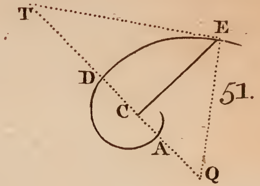

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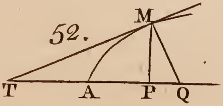

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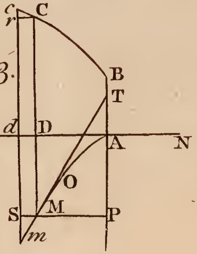

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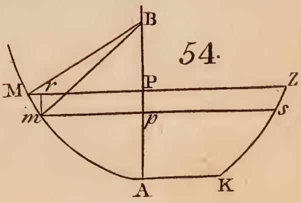

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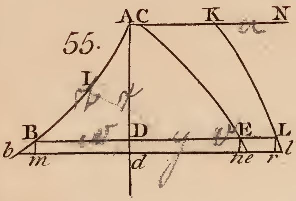

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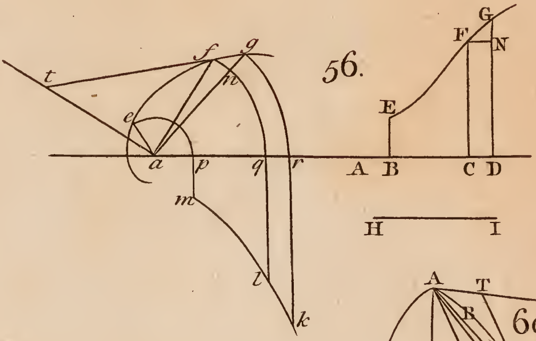

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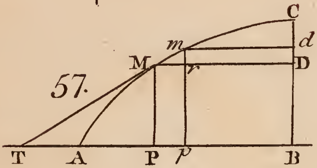

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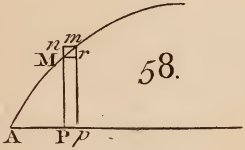

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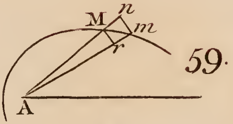

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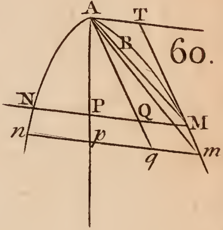

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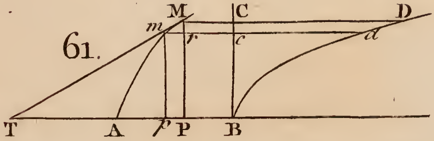

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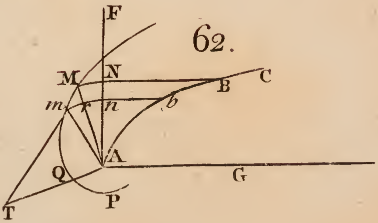


PROP. XIII.

Let AMB *be any curve,* AP *its axis,* PM *an ordinate; and let the curve* ANE *be of ſuch a nature, that producing* PM *to* N, *the tangent* NT *may be parallel to the chord* AM. *Then drawing* AD, ND *parallel to* MP, AP; *the area* ADN *will be equal to the area* APM. 66.

Let m be a point infinitely near M, and draw pmn parallel to PM, and dn, NS, Mr parallel to AP. Then the triangles Nns, TNP, and AMP are ſimilar, whence $AP : PM :: TP : PN :: Ns : sn$. Therefore $AP \times sn = PM \times Ns$; that is, $DN \times sn = PM \times Mr$, or the trapezium DN$nd$ = trapezium PMmp. And the ſum of all the trapeziums DNnd = ſum of all the trapeziums PMmp; that is, the area ADN = area APM.

PROP. XIV.

If AMB *be any curve,* AP *the axis,* PM *an ordinate; and let* KZL *be ſuch a curve, that taking any right line* r, *and continuing the ordinate* MP *to* Z, *it may be,* $\sqrt{\textit{area}\ \text{APM}} : \text{PM} :: r : \text{PZ}$. *Then the area* APZK $= 2r \times \sqrt{\textit{area}\ \text{APM}}$. 67.

Take m infinitely near M, and draw mpz parallel to MP. And put z = area APM = ſum of the trapezia P$pm$M or $PM \times Pp$, $AP = x$, $PM = y$, $PZ = v$; then $\sqrt{z} : y :: r : v = \frac{ry}{\sqrt{z}}$, and $v \times Pp$ or $PZ \times Pp = \frac{ry}{\sqrt{z}} \times Pp = \frac{r}{\sqrt{z}} \times PM \times Pp = \frac{r}{\sqrt{z}} \times$ trapezium PMmp. Let w = trapezium PMmp.

Fig. 67. PM*mp*, and divide the area APM or z into an infinite number of trapeziums equal to $w = 1$. Then all theſe areas denoted by z conſtitute a ſeries of quantities in arithmetic progreſſion, and the ſum of all the roots $\frac{1}{\sqrt{z}}$, (by Prop. V. Arithm. of Infinites) is $\frac{1}{m+1} \times$ ſo many times the greateſt term, where $m = -\frac{1}{2}$, the greateſt term $\frac{1}{\sqrt{z}}$, and number of terms z, therefore the ſum of all the $\frac{1}{\sqrt{z}}$, is $= \frac{1}{1-\frac{1}{2}} \times \frac{z}{\sqrt{z}} = 2\sqrt{z}$. And therefore ſince $PZ \times Pp = \frac{r}{\sqrt{z}} \times w$. Therefore the ſum of all the $PZ \times Pp = r \times$ ſum of all the $\frac{w}{\sqrt{z}}$ or $r \times$ ſum of all the $\frac{1}{\sqrt{z}} = r \times 2\sqrt{z}$; that is, the area $APZK = r \times 2\sqrt{\text{area APM}}$.

PROP. XV.

68. *Let* AMB *be any curve*, AD *the axis*, PM *an ordinate*, MT *a tangent. And let the curves* EX *and* GY *be ſuch, that continuing* MP *to* X, *and drawing* MQY *parallel to* AP; *it be* TP : PM : : QY : PX. *Then the area* APXE = *area* DGQV.

Take the point m infinitely near M, and draw mpx parallel to MP, and mqy parallel to MQY. Then the triangles M*rm*, TPM are ſimilar, whence M*r* : *rm* : : TP : PM : : (conſtruction) QY : PX; therefore $Mr \times PX = rm \times QY$, or $Pp \times PX = Qq \times QY$; that is, the trapezium PX*xp* = trapezium QYyq. And the ſum of all the trapezia PXxp = ſum of all the trapezia QYyq; that is, the area APXE = area DGYQ. Fig. 68.

PROP. XVI.

69. *Suppoſe* AMB *any curve*, DM *an ordinate*; DT *perpendicular to it*, MT *a tangent. And let the curves* EN, *and* GH *be of ſuch a nature, that producing* DM *to* N, *and with radius* DM *and center* D, *deſcribing the arch* MrQ, *and drawing* QH *and* DG *perp. to* DQ. *Then making* TD : DM : : DM × QH : DN^2. *Then the area* DGHQ = *twice the area* EDN.

Take the point m infinitely near M, and draw $drmn$. From the center D, with radius DN deſcribe the ſmall arch Ns; and with the radius Dm, the arch mq, and draw qh parallel to QH. Then the triangles Mrm and TDM are ſimilar, whence Mr : rm : : TD : DM : : (conſtruction) DM × QH : DN^2; therefore Mr × DN^2 = rm × DM × QH. But the ſectors DMr, DNs are ſimilar, whence DM : Mr : : DN : Ns, and DM × Ns = DN × Mr, and DN × DM × Ns = DN^2 × Mr = rm × DM × QH; and dividing, DN × Ns = rm × QH = Qq × QH; that is, twice the triangle DNn = trapezium QHhq. Therefore the ſum of all theſe trapezia or the area DGHQ = twice the ſum of all the triangles, or twice the area DEN.

Fig.

PROP. XVII.

70. *Let* DOM *be any curve,* DA *a tangent at* D, DM *an ordinate,* DT *perp. to it,* MT *a tangent. And let* AGF *be ſuch a curve, that producing* DM *to* F. *And taking* $a =$ *any given line, let* $DT : 2a :: DM^2 : DF^2$. *Then the area* $ADF = a \times DM$.

Take the point m infinitely near M, and draw Dmf; alſo draw Mr perp. to Df, and fn perp. to DF. Then the triangles MDT, mrM are ſimilar, and therefore $DT : DM :: Mr : mr$, whence $DT \times mr = DM \times Mr$. But $DM \times Mr : DF \times nf ::$ triangle DMm : triangle $DFf :: DM^2 : DF^2 ::$ (conſtruction) $DT : 2a$. Therefore $2a \times DM \times Mr = DT \times DF \times nf$, or $2a \times DT \times mr = DT \times DF \times nf$, whence $2a \times mr = DF \times nf =$ twice the triangle DFf. And $a \times mr = DFf$; therefore $a \times$ all the $mr =$ ſum of all the triangles DFf; that is, $a \times DM =$ area DAF.

PROP. XVIII.

61. *If* AM *be any curve,* AP *the abſciſſa,* PM *an ordinate,* BC *parallel to it,* MT *a tangent at* M. *And if* BD *be ſuch a curve, that drawing* MCD *parallel to* AP, CD *may be always equal to the ſubtangent* TP. *And both curves* APM *and* BCD *revolve about the axis* APB. *Then the ſolid deſcribed by* BCD, *will be double the ſolid deſcribed by* APM.

Let m be a point of the curve, infinitely near M, and draw mp parallel to MP, and md parallel to AP. Then the triangles mrM, TPM are ſimilar; whence $mr : Mr :: TP : PM$; that is, $Pp : Cc :: CD : PM$; therefore $Pp \times PM = Cc \times CD$. Put $p =$ circumference deſcribed by M or D; then the area deſcribed by Cc is $p \times$ Cc, and the area Fig. 61. deſcribed by PM $= \frac{p \times \mathrm{PM}}{2}$. And the ſolid deſcribed by C$cd$D is $p \times$ C$c \times$ CD, and the ſolid deſcribed by P$pm$M is $\frac{p \times \mathrm{P}p \times \mathrm{PM}}{2} = \frac{p \times \mathrm{C}c \times \mathrm{CD}}{2}$. Therefore the ſolid deſcribed by C$cd$D is double the ſolid deſcribed by P$pm$M. And the ſum of all the C$cd$D or the ſolid CBD is double the ſum of all the P$pm$M or the ſolid APM.

PROP. XIX.

Let AFM *be any curve,* AP *the axis,* PM *an ordinate,* MC *perpendicular to the curve in* M. *And if* BOD *be ſuch a curve, that producing* MP *to* D, PD *may be equal to* MC. *Then if the curve* AFM *revolve about the axis* AP, *the ſurface deſcribed by* AFM *will to the area* APDB, *as the circumference of a circle to the radius.* 63.

Take Mm infinitely ſmall, and draw mpd parallel to MP, and Mr parallel to AP. Then the triangles MPC and Mrm are ſimilar; whence Mr : Mm : : MP : MC; therefore M$r \times$ MC $=$ M$m \times$ MP. Let $p =$ circumference deſcribed by M; then the area deſcribed by Mm, will be $p \times$ Mm. And the area P$pd$D $=$ P$p \times$ PD $=$ M$r \times$ MC. Therefore the ſurface deſcribed by Mm : area P$pd$D : : M$m \times p$: M$r \times$ MC or M$m \times$ MP : : p : MP : : circumference : radius. And the ſum of all the ſurfaces Mm : ſum of all the P$pd$D; that is, the ſurface deſcribed by AFM : area APDB : : circumference : radius.

Fig.

PROP. XX.

61. *If* AM *be any curve*, AP *the axis*, PM *an ordinate*, BC *parallel to it*, MT *a tangent at* M. *If* MD *be drawn parallel to* AB; *and the curve* BD *be ſuch, that* CD *be always equal to the tangent* MT. *Then the ſurface deſcribed by the curve* AM *revolving about the axis* AP, *is to the area* BCD, *as the circumference of a circle, to the radius.*

Take the part M*m* infinitely ſmall, and draw *mp* parallel to MP, and *md* parallel to MD; and let $p =$ circumference to the radius PM. The triangles *mr*M, TPM are ſimilar, whence M*r* : M*m* : : MP : MT or CD; therefore $Mr \times CD = Mm \times MP$. Then the ſurface deſcribed by $Mm = Mm \times p$, and the area $CcdD = Cc \times CD = Mr \times CD = Mm \times MP$. Therefore the ſurface deſcribed by M*m* : area C*cd*D : : $Mm \times p$: $Mm \times MP$: : p : MP. And the ſum of all the ſurfaces M*m* : ſum of all the areas C*cd*D : : p : MP; that is, the ſurface deſcribed by the curve AM : area of the curve CBD : : circumference : radius.

SECT. II.

Of the Axes, Ordinates and Aſſymptotes of Curves, their nature and properties in general; their affections deduced from the equations of the curve; the product of the ſegments of parallel lines; generation of curves by ſhadows.

PROP. I.

If the abſciſſa of a curve be repreſented by x, *and the ordinate by* y; *and the nature of the curve be expreſſed by any equation between* x *and* y. *Then if any particular value be aſſigned to* x, *there will be as many values for* y, *as is the index of its higheſt power in the equation, and no more.* Fig.

Suppoſe any equation, as $y^3 - aay + xy^2 - axx = 0$, and put $b = x$, then the equation will become $y^3 - aay + by^2 - abb = 0$, which is a cubic equation with the unknown quantity y, and therefore by the principles of algebra, this equation has three roots, which are the three values of y correſponding to that value of x denoted by b. And aſſigning any other value for x, as ſuppoſe c; then there will be produced another cubic equation, having three roots or values of y correſponding to c; and ſo on for any other value of x. And the ſame reaſoning holds good for any other equation how high ſoever it be, that there are as many va-

Fig. lues of y, as is the dimenſion of the higheſt power of it. And there can be no more, becauſe there are no more roots of the equation, than the higheſt power of the unknown quantity has dimenſions.

Cor. 1. *In an equation between x and y, expreſſing the nature of a curve; for any particular value of y; there are as many values of x, as the higheſt power of it has dimenſions in the equation.*

For in the former equation, put $b = y$, then $b^3 - aab + bbx - axx = 0$, a quadratric equation for the unknown quantity x, which equation has two roots or values of x. And the like for other equations.

Cor. 2. *Among the ſeveral values of y, for any determinate value of x; thoſe that are affirmative are drawn on the upper ſide of the abſciſſa to the curve; and thoſe that are negative, are drawn from the abſciſſa downwards to the curve.*

And among the values of x, for any particular value of y, if thoſe that are affirmative, lie from the origin of the abſciſſa to the ordinate, on the right; then thoſe that are negative, will lie from the origin of the abſciſſa to the ordinate, on the left hand.

For affirmative and negative quantities muſt lie contrary ways from any fixed point or given line.

Cor. 3. *When ſuch a value of x is aſſigned, that in the reſulting equation, ſome of the roots are impoſſible. This denotes that ſuch ordinates as are repreſented by theſe impoſſible roots, don't reach the curve; and in that caſe there is no ſuch ordinate.*

For impoſſible roots in an equation always ſhew the impoſſibility of the thing propoſed.

Cor. 4. *Hence by aſſigning ſeveral values of x, you may determine as many values of y as you pleaſe, and by that means find as many points of the curve as* *as you pleaſe; which being drawn thro' all theſe points will be the* LOCUS *of that equation.* Fig.

Cor. 5. *In aſſigning any value for x, if one root or value of y be* 0, *then the curve cuts the abſciſſa in that point. But if y be infinite, then the ordinate to that point, is an aſſymptote to the curve, which runs out ad infinitum.*

Cor. 6. *If x be made infinite, and then ſome value of y is* 0, *this ſhews that x or the abſciſſa is an aſſymptote to ſome branch of the curve, which runs on infinitely.*

PROP. II.

If a curve is defined by an equation ſhewing the relation of the abſciſſa (x), and ordinate (y); a right line may cut that curve in as many points, as the higheſt power of y has dimenſions in that equation, and not in more points. 71.

Let RAMN be a curve defined by the equation $y^3 - aay + by^2 - ax^2 = 0$. It is ſhewn in the laſt Prop. that for any determined value of AP or x, there will be three values of y, viz. PM, PN, and PR, becauſe the equation has three roots or values of y. Therefore the right line NR cuts the curve in the three points N, M, R. But there being no more roots or values of y, it cannot cut the curve in more points. It may cut it in leſs, when ſome of the roots are impoſſible. As if PM was to move into the poſition QO, where it touches the curve at O, there the two points M, N, coincide; but beyond Q, the ordinate cannot touch that part of the curve at all, and then the two roots PM, PN vaniſh.

Cor. 1. *Hence one root alone cannot vaniſh out of the equation, or one ordinate vaniſh out of the curve; but two muſt vaniſh together.*

Fig. 71. For firſt the two ordinates PM, PN, coincide with QO; and then vaniſh together, and are no where found.

Cor. 2. *In a line of the third, fifth, &c. or of any order whoſe dimenſion is an odd number; the abſciſſa, or any right line not parallel to an aſſymptote, will cut the curve at leaſt in one point.*

For if the equation of the curve be odd, it will at leaſt have one real root.

Cor. 3. *If a right line cut a line of the third order in two points, it will neceſſarily cut it in three.*

For if the equation of the curve has two roots real, it will have three.

Cor. 4. *If a right line touch a line of the third order; it will neceſſarily cut the curve in ſome point.*

For the point of contact muſt be taken for two points which coincide together.

Cor. 5. *If a right line touch a line of the third order in the point of contrary flexure; it will cut the curve there, and in no other point.*

For three points muſt be ſuppoſed to coincide there.

Cor. 6. *If a right line cut a curve in a double point of the curve, this muſt be taken for two points. And a line that touches one arch of the curve in that point; cuts the other in the ſame point.*

PROP. III.

Every geometrical curve being continued, either returns into itſelf, or goes on to an infinite diſtance.

Any geometrical curve may be conſidered as deſcribed by the continual motion of a point; and this being effected by ſome certain law, the motion Fig. muſt needs continue *ad infinitum*. And therefore 71. the ſpace run thro', that is, the line itſelf, muſt either return back or proceed on infinitely, by reaſon of the regular motion of the deſcribing point.

Cor. 1. *If any curve has two infinite branches or legs, they join one another either at a finite, or at leaſt at an infinite diſtance.*

For the curve being deſcribed by the continual motion of a point, it muſt needs paſs out of one branch into another.

PROP. IV.

Let BM *be any curve,* AP *the abſciſſa,* PM *an* 72. *ordinate. If inſtead of* MP, MQ *and the parallels thereto be made ordinates; the equation between* AQ *and* QM, *denoting the nature of the curve, ſhall not riſe to higher dimenſions, than the equation between* AP *and* PM, *did before.*

Let AP $= x$, PM $= y$, AQ $= v$, QM $= w$. Then ſince the lines MP, MQ, QP are given in poſition, their ratio to one another is given; therefore let PM $= rw$, and QP $= sw$; then AP $= v + sw$, PM $= rw$. So now we ſhall have the equation of the curve made up of $v + sw$, and rw, which before was made up of x and y. It is plain if $v + sw$ be ſubſtituted for x, and rw for y, it will ariſe to no higher powers of v and w, than it did before of x and y, becauſe v and w are but of one dimenſion; for $v + sw$ raiſed to any power, can conſiſt of terms of no higher power than the like power of x. So that now the curve will be defined by an equation where the higheſt powers of v and w, are the ſame as the higheſt powers of x and y in the firſt equation

Fig. 72. Cor. *If the abſciſſa begin at any other point* C, *ſtill the equation between* CQ *and* QM *will riſe but to the ſame hight as that between* AP *and* PM.

For the abſciſſa is now increaſed or leſſened by ſome given quantity as a; therefore if we put $v = CQ$, then $AP = a + v + sw$; the equation ſtill contains only one dimenſion of v and w, and therefore can riſe to no higher power than before.

PROP. V.

73. *An aſſymptote may interſect the curve in ſo many points abating two, as the equation of the curve has dimenſions, and no more.*

A right line may interſect a curve in as many points as the equation of the curve has dimenſions, and no more, by Prop. II. Suppoſe then that DAE is a line of the third order, whoſe aſſymptote is CB. And let the right line DFE cut it in the three points D, L, E; and the aſſymptote in F. Let the line DE revolve about the pole F, ſo as that the point D may approach towards C; by this motion the points D, E of interſection, will move further and further from the point F, till at laſt when the line DE falls upon the aſſymptote CB, the points of interſection with the curve, D and E are removed to an infinite diſtance; and there remains only the point A, in which DE or the aſſymptote CB, can cut the curve. And in the ſame manner it is ſhewn in any other caſe, that two points of interſection go off to an infinite diſtance. And therefore the number of points in which an aſſymptote can cut the curve, will want two, of the number of dimenſions, in the equation of the curve.

Cor. *In a conic ſection, which is the ſecond gender of lines, the aſſymptote does not cut the curve at all.*

In the third gender it can only cut it in one point. In Fig. 73. *the fourth gender in two points, and no more. And ſo forward.*

PROP. VI.

74. *Lines of the third, fifth, ſeventh, order, &c. or any odd number, have at leaſt two infinite legs or branches, running contrary ways.*

Suppoſe NAM a curve of ſuch an order; then ſince this is an equation of ſome odd power, therefore from the nature of equations in algebra, this equation muſt at leaſt have one real root, whatever the value of AP or x be; therefore ſuppoſe AP infinite towards P, then PM will be a root of the equation, and have a real value, and M will be a point of the curve. And the branch AM will be infinite. Again, let AB be $= - x$, the equation ſtill will have one real root BN, and N will be another point of the curve; and the branch AN will be infinite. So there will be the two branches AM and AN both infinite. And perhaps there may be more branches infinite upon other conſiderations.

Cor. 1. *Hence the roots* PM *or* BN *may be infinite; in which caſe the curve runs off infinitely from the abſciſſa* BP. *Or they may be finite; in which caſe* AP *or* AB *may be an aſſymptote to the curve.*

Cor. 2. *In lines of the ſecond, fourth, ſixth, or any even number of dimenſions; the roots of the equation may be all impoſſible; and then the figure may return into itſelf, and be contained within certain bounds.*

PROP. VII.

75. *The affections of a curve may be known from the equation of that curve.*

Let the abſciſſa AP $= x$, the ordinate PM $= y$; then the affections of the curve may be traced out, by putting x or y, ſucceſſively $= 0$, and infinity, and equal to any of the given coefficients, affirmative or negative, &c. By ſuch equations as reſult, all the points of the curve will be determined, and where it cuts the axis, the poſition of the aſſymptotes, if it have any, and ſo on.

For example, let $ay^2 - xyy = x^2 + bx^2$, or $\overline{a-x} \times yy = \overline{b+x} \times xx$, where $a =$ AD, $b =$ AB; then $yy = \frac{b+x}{a-x} \times xx$, and $y = \pm x \surd \frac{b+x}{a-x}$. This ſhows that any ordinate may be either affirmative or negative, and may be taken both above and below the abſciſſa, as PM, Pm, and equal to one another. Put $x = 0$, then $y = 0$, this ſhows that the curve cuts the axis at A the beginning of the abſciſſa. Put $x = a =$ AD, then $y = a \surd \frac{b+a}{a-a} = a \surd \frac{b+a}{0} =$ infinity; therefore the ordinates erected at D will be infinite, or EDF will be an aſſymptote to the curve. If x be greater than a, then $\surd \frac{b+x}{a-x}$ will be the ſquare root of a negative quantity; which ſhews that no part of the curve can run beyond the line or aſſymptote EF. If x lie the contrary way; that is, if x be negative, then $y = \mp x \surd \frac{b-x}{a+x}$, now if x be leſs than b, then y will have two equal values, which are equal to one another; and that which Fig. 75. was the affirmative one before, is the negative one now; and what was negative before is now affirmative, cauſed by changing $\pm x \surd$ to $\mp x \surd$. But here ſuppoſe $x = b$, then $y = \mp x \sqrt{\frac{b-b}{a+b}} = 0$; and therefore the curve will paſs thro' the point B, and be a tangent to the ordinate. If x be greater than b, then $y = \mp x \sqrt{\frac{b-x}{a+x}}$ becomes an impoſſible quantity, ſo that the curve goes no further to that hand than to D. If we make $y = 0$; that is, $x \sqrt{\frac{b-x}{a+x}} = 0$; then $xx \times \overline{b+x} = 0$, here the roots are $x = -b$, $x = c$, $x = 0$. Therefore the curve paſſes twice thro' the point A, and once thro' the point B; and therefore A is a *double point* of the curve.

Suppoſe $b = 0$, then the points B and A coincide, and the *nodus* or part of the curve between A and B vaniſhes. Here $y = x \sqrt{\frac{x}{a-x}}$, and $\overline{a-x} \times yy = x^3$; and this is an equation to the ciſſoid, by Cor. II. Prop. I. Sect. II. Book I. which is plain by comparing their equations together; x repreſenting there what y repreſents here; and y repreſenting there what x does here; and $2a$ there the ſame as a here.

Put x = infinity, then $\overline{a-x} \times yy = \overline{b+x} \times xx$, becomes $-xyy = x^3$, or $-yy = xx$, and $x = \sqrt{-yy}$, an impoſſible equation; and therefore x cannot be infinite.

Put y = infinity, then $\pm x \sqrt{\frac{b+x}{a-x}}$ = infinity, and $xx \times \frac{b+x}{a-x}$ = infinity; therefore ſince x is finite, Fig. 75. finite, we ſhall have $a - x = 0$, or $x = a =$ AD, and DE an infinite ordinate, or an aſſymptote, as before.

If we ſuppoſe $a = b$; this makes no alteration in the nature of the curve.

PROP. VIII.

In any curve, if tangents be drawn to all points of the curve; and if they always cut the abſciſſa at a finite diſtance from the beginning of it; that curve has an aſſymptote; otherwiſe not.

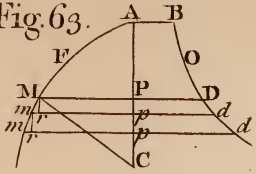
76. Let A*m*M be any curve, AP the axis, not being an aſſymptote, or parallel to one. To all the points *m*, M, ſuppoſe tangents to be drawn as *mt*, MT, cutting the abſciſſa in *t* and T. Now if *m* be any point of the curve near the vertex, *mt* its tangent, and if A*t* be a finite quantity; and if *m* be ſuppoſed to move further and further from the vertex, to M, and that the diſtance AT of the tangent MT is ſtill finite; and if M be removed to an infinite diſtance, the length AT ſtill continues of a finite magnitude. Then the tangent MT will be an aſſymptote to the curve. For an aſſymptote is nothing elſe than a tangent drawn to a point of a curve infinitely diſtant. But if T be at an infinite diſtance, then ſuch a tangent is no where to be found, and therefore in ſuch a caſe the curve has no aſſymptote.

For example, let AM be a parabola; then if AM be infinite, AP is infinite; and AT = AP is alſo infinite. Therefore AM has no aſſymptote.

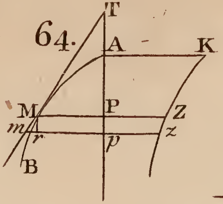
If AM be an hyperbola, and M at an infinite diſtance; then the tangent MT cuts the axis AP in the center T; then AT is half the tranſverſe axis; conſequently MT is an aſſymptote.

Cor.

Fig. 63.



64.



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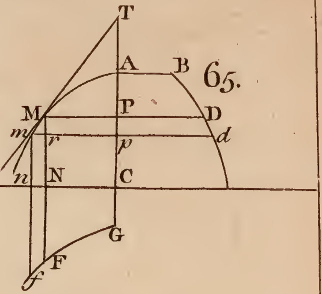

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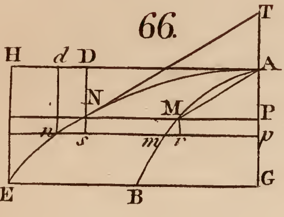

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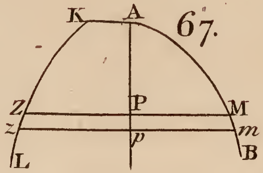

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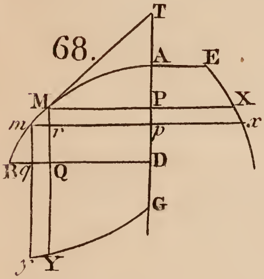

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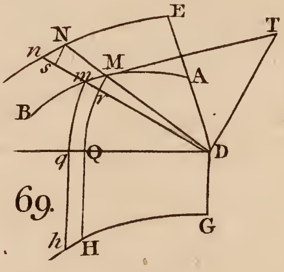

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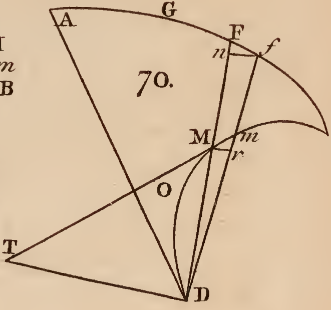

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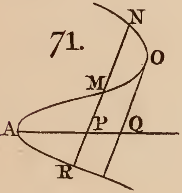

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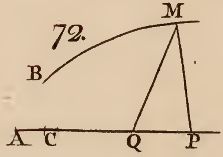

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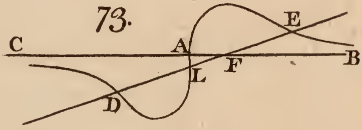

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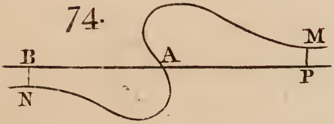

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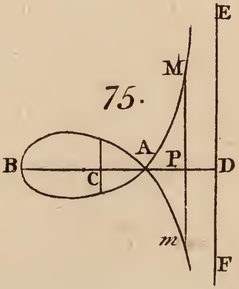

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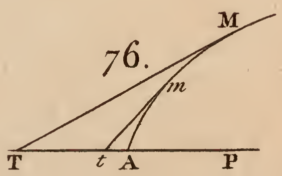


Cor. *Hence to find the aſſymptote to a curve, is the ſame thing as finding the tangent to a point of it, at an infinite diſtance.* Fig. 76.

PROP. IX.

If the equation of a curve be $y = a + bx + cx^2 + dx^3$, *&c. to* fx^n. *And n be an even number; then the curve has two infinite legs both on the ſame ſide of the abſciſſa* AP. 77.

For when x is infinite, then $y = fx^n$, all the other terms vaniſhing; ſo that if $y = fx^2$, or $y = fx^4$, or $y = fx^6$, &c. and ſuppoſe x infinite, then whether AP be $= + x$ or $- x$, ſtill $y = + fx^2$, or $y = + fx^4$, or $y = + fx^6$, &c. that is, the value of y will always be affirmative, when n is even; and therefore the curve is always on the ſame ſide of the abſciſſa.

For example, if a, b, c, d be $= 0$, and $n = 4$, then $y = fx^4$, and AP $= x$, then PM $= fx^4$, and if Ap $= - x$, then again $pm = + fx^4$, and therefore both the parts AM, Am, infinitely extended, lie on the ſame ſide of the abſciſſa AP. This curve is a kind of parabola.

PROP. X.

If the equation of the curve be $y = a + bx + cx^2 + dx^3$, *&c. to* fx^n. *And n be an odd number. Then the curve has two infinite legs, running contrary ways, and on different ſides of the abſciſſa.* 78.

For when x is infinite, all the terms of x vaniſh except the greateſt fx^n; therefore $y = fx^n$. But if $n = 1, 3, 5, 7$, &c. and the abſciſſa $= - x$, then Pm or the ordinate, $y = - fx$, or $y = - fx^3$, or $y = - fx^5$, &c. that is, if x be negative, and n an

Fig. 78. n an odd number, then $y = -fx^n$, whence the ordinate lies on the contrary ſide of what it does when it is $+x$, for then $y = +fx^n$. And the branches of the curve AM, Am run contrary ways, becauſe $-x$ is contrary to $+x$; that is, the abſciſſas lie different ways.

For example, let $y = x^3$. Then if AP $= x$, then $y = fx^3 =$ PM; but if A$p = -x$, then $y = -fx^3 = pm$, on the contrary ſide of AP. So the curve MAm has two infinite branches running contrary ways, and on different ſides of AP. This curve is the cubical parabola.

PROP. XI.

79. *If the equation of the curve be* $y = \frac{a^{n+1}}{x^n}$; *this curve has four infinite legs contained between two aſſymptotes; which lie in the oppoſite angles* DAC, BAE, *when n is an odd number; but in the adjacent angles*, DAC, DAB, *when n is even.*

Let AP $= x$, PM $= y$, ſuppoſe x affirmative, then PM $= \frac{a^{n+1}}{x^n}$; if x be $= 0$, then PM is infinite, and the curve runs on infinitely towards D; therefore DA is an aſſymptote. And if x be infinite, then PM $= 0$, and the curve runs on infinitely towards C, and AC is an aſſymptote.

Let x be negative, or AQ $= x$, then if n is an odd number, $y = -\frac{a^{n+1}}{x^n}$; therefore y or QN lies on the contrary ſide of BC, and the curve is contained in the angle BAE. Let $x = 0$, y is infinite, and the curve runs on infinitely toward E.

And if $x = -$ infinity, then $y = 0$, and the curve Fig.
runs on ad infinitum towards B. 79.

But whilſt x is negative, if n be an even number, then $y = + \frac{a^{n+1}}{x^n}$; and therefore y or Qm lies on the ſame ſide of BC as at firſt. And as x is $= 0$ or infinity, y will be infinite or nothing; therefore the curve runs infinitely towards B and D, and is contained in the angle BAD. So that there are four infinite branches of the curve, contained between the two aſſymptotes BC, DE; either in the oppoſite angles, when n is odd; or in the adjacent angles, when n is even. And here one of the aſſymptotes AC is the abſciſſa of the curve. Theſe are called *Hyperbolic Curves.*

Cor. *If* $y = \frac{aa}{x}$, *or* $xy = aa$, *then the curve is the common hyperbola, and ſince* $n = 1$, *therefore the oppoſite ſections are contained in the oppoſite angles* DAC, BAE; *between the aſſymptotes* BC, DE, *as is commonly known.*

PROP. XII.

If FM *be any curve,* FD *the diameter; and if* 80.
an ordinate MD *be drawn to cut the curve in as many points as it has dimenſions; then the ſum of the ordinates on one ſide* FD *is equal to the ſum of the ordinates on the other ſide; taken in the ſame line* MG.

Let the nature of the curve be denoted by this equation; $y^n + \overline{ax + b} \times y^{n-1} + \overline{cxx + dx + e} \times y^{n-2}$ &c. $= 0$, (ſee Schol. to the definitions), and this general equation referred to ſome abſciſſa AP, and ordinate PM. Produce AP to C till AC $= \frac{b}{a}$, and draw AB parallel to

Fig. 80. MP and $= \frac{b}{n}$. Thro' C, B, draw CFBD, and take BD for a new abſciſſa, and DM its ordinate; and let BD $= v$, DM $= z$. Take $1 : p ::$ BD (v) : AP $= pv = x$. By ſimilar triangles, CA : AB :: CP : PD; that is, $\frac{b}{a} : \frac{b}{n} :: pv + \frac{b}{a}$: PD $= \frac{apv + b}{n}$, and PM $=$ MD $-$ PD $= z - \frac{apv + b}{n}$, or $y = z - \frac{apv + b}{n}$. And involving, $y^n = z^n - \overline{apv + b} \times z^{n-1} + \overline{n - 1} \times \overline{apv + b}^2 \times z^{n-2}$ &c. and $y^{n-1} = z^{n-1} - \overline{n - 1} \times \frac{apv + b}{n} \times z^{n-2}$ &c. $y^{n-2} = z^{n-2}$ &c. ſubſtitute theſe values of y, and x in the equation $y^n + \overline{ax + b} \times y^{n-1} + \&c. = 0$, and we have

$$z^n - \overline{apv + b} \times z^{n-1} + \overline{n - 1} \times \overline{apv + b}^2 \times z^{n-2} \text{ \&c.}$$

$$+ \overline{apv + b} \times z^{n-1} - \overline{n - 1} \times \frac{\overline{apv + b}^2}{n} \times z^{n-2} \text{ \&c.}$$

$$+ cppvv \times z^{n-2} \text{ \&c.}$$

Here the ſecond term of the equation vaniſhes, and therefore the ſum of the affirmative roots or ordinates is equal to the ſum of the negative ones, by the nature of equations; that is, the ſum of the ordinates on one ſide is equal to the ſum on the other ſide of BD, and BD is the diameter of the figure.

Cor. 1. *In curves of the firſt gender, or the conic ſections; the ordinates are biſſected by the diameter.*

Cor. 2. *In curves of the ſecond gender; the diameter cuts all the ordinates; ſo that the ſum of two parts on one ſide is equal to the third part on the other ſide.* Fig. 80.

Cor. 3. *The line which cuts two parallels, that the ſum of the parts on one ſide be equal to the ſum of the parts on the other; will cut all other parallels in the ſame manner.*

PROP. XIII.

If a right line FN *cut a curve of any order, in the points* M, *m*, N, *&c. and its aſſymptotes in* F, C, G. *Then the parts* FM, GN, *lying the ſame way from their aſſymptotes to the curve, will be equal to the part or parts* C*m*, *lying the contrary way from their aſſymptotes to the curve.* 81.

Let AP be the diameter to the ordinate PM, and let the curve be a line of the third order. Then (Prop. XII.) $PM + Pm = PG$. Now becauſe the curve coincides with the aſſymptotes at an infinite diſtance; therefore M, F; and *m*, C; and N, G coincide. But at an infinite diſtance it is alſo (Prop. XII.) $PM + Pm = PN$; that is, $PF + PC = PG$. But if this be ſo at an infinite diſtance, it will be the ſame at any other diſtance; as parallel lines are cut in the ſame proportion; therefore it will be in general $PC + PF = PG$, but $Pm + PM = PN$; therefore taking equals from equals, $Pm - PC + PM - PF = PN - PG$; that is, $Cm - FM = GN$, or $Cm = FM + GN$.

And the ſame is eaſily ſhewn of curves of higher orders.

Fig.

PROP. XIV.

82. *If* ANB *be a curve of the firſt order*, AB *a right line cutting the curve in* A *and* B; MN *another right line, cutting the curve alſo in* M *and* N, *and the other in* P. *Then the rectangle* APB, *will be to the rectangle* MPN *in a given ratio, having any parallel poſition.*

Let AP be the abſciſſa, PM and PN ordinates, and let AP $= x$, PM or PN either of them $= y$. Then the equation of the curve (Schol. Def.) in general, is $y^2 + \overline{ax + b} \times y + cxx + dx = 0$; in which the determinate quantity e is 0, becauſe the abſciſſa begins at the curve. Let $y = 0$, then $cxx + dx = 0$, becauſe all the terms vaniſh where y is found, then $cx = -d$, and $x = -\frac{d}{c} =$ AB; therefore AB — AP or PB $= -\frac{d}{c} - x$, or rather $x + \frac{d}{c}$, and AP × PB $= xx + \frac{dx}{c}$. But by the nature of equations, the laſt term in which the unknown quantity is wanting, is the product of all the roots; that is, $cxx + dx =$ PM × PN. Therefore AP × PB : PM × PN $: : xx + \frac{dx}{c} : cxx + dx : : 1 : c$; but whilſt the paralleliſm of the lines AB, MN continues, the quantity c remains the ſame; and conſequently the proportion of theſe rectangles remains the ſame.

PROP.

PROP. XV.

83. *Let* AMN *be a line of the third or any other order; and let the right line* AD *cut it in the three points* A, B, D; *and the line* MN *in three others* M, *m*, N; *and let* P *be their point of interſection. Then will* PA × PB × BD *be to* PM × P*m* × PN *in a given ratio; whilſt they are in any poſition parallel to* AD, MN.

Let the abſciſſa AP $= x$, and any of the lines PM, P*m*, PN $= y$. Then (Schol. definitions) the nature of the curve is denoted by this equation, $y^3 + \overline{ax + b} \times y^2 + \overline{cx^2 + dx + e} \times y + fx^3 + gx^2 + hx = 0$, where the determinate quantity k is $= 0$, becauſe the abſciſſa begins at the curve. Let $y = 0$, and all the quantities will vaniſh where y is found, therefore $fx^3 + gx^2 + hx = 0$. In which caſe, we ſhall have $x = \frac{-g}{2f} \pm \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$; that is, AB $= \frac{-g}{2f} - \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$, and AD $= \frac{-g}{2f} + \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$; therefore BP $=$ AP $-$ AB $= x + \frac{g}{2f} + \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$, and PD $=$ AD $-$ AP $= -\frac{g}{2f} + \sqrt{\frac{gg}{4ff} - \frac{h}{f}} - x$, or rather $x + \frac{g}{2f} - \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$. Therefore AP × BP × DP $= x \times \overline{x + \frac{g}{2f} + \sqrt{\frac{gg}{4ff} - \frac{h}{f}}} \times \overline{x + \frac{g}{2f} - \sqrt{\frac{gg}{4ff} - \frac{h}{f}}} = x \times \overline{x + \frac{g}{2f}}^{2} - x \times$

Fig. 83. $\overline{\frac{gg}{4ff}-\frac{h}{f}} = x \times \overline{xx+\frac{gx}{f}+\frac{gg}{4ff}-\frac{gg}{4ff}+\frac{h}{f}} = x \times \overline{xx+\frac{gx}{f}+\frac{h}{f}} = x^3+\frac{gx^2}{f}+\frac{hx}{f}$. But by the nature of equations, the laſt term is equal to the product of all the roots; that is, MP × mP × NP $= fx^3+gx^2+hx$. Therefore AP × BP × DP : MP × mP × NP : : $x^3+\frac{gx^2}{f}+\frac{hx}{f} : fx^3+gx^2+hx$ 1 : f. But whilſt the lines AD, MN, have any other places parallel to the former; the quantity f remains the ſame; and therefore the ratio of AP × BP × DP to MP × mP × NP.

Otherwiſe thus,

Let R be the beginning of the abſciſſa, which cuts the curve in the three points A, B, D; and put RP $= x$, and let the ordinate MP cut the curve in the three points M, m, N. Then in the general equation, when y vaniſhes, we ſhall have $fx^3+gx^2+hx+k=0$, and $x^3+\frac{g}{f}x^2+\frac{h}{f}x+\frac{k}{f}=0$, the roots of which equation are RA, RB, RD; therefore the equation is made up of the three quantities $x-$ RA, $x-$ RB, $x-$ RD, multiplied into one another. Whence $x^3+\frac{g}{f}xx+\frac{h}{f}x+\frac{k}{f} = \overline{x-RA}\times\overline{x-RB}\times\overline{x-RD} = \overline{RP-RA}\times\overline{RP-RB}\times\overline{RP-RD}$ = AP × BP × — PD. But in any equation, the laſt term is equal to the product of all the roots. Therefore $fx^3+gx^2+hx+k=$ MP × mP × — NP. Therefore the product of AP × BP × — PD : product of MP × mP × NP; or AP ×

BP × PD : MP × mP × NP :: $x^3 + \frac{g}{f}xx + \frac{h}{f}x$ Fig. 83.
$+ \frac{k}{f} : fx^3 + gxx + hx + k :: 1 : f$.

It is evident, the ſame way of reaſoning may be applied to a curve of any gender, and therefore when the angle of inclination APM is given; if the right lines AP, PM; or RP, PM, cut a curve line of any order in as many points as it has dimenſions; it will always be, that the product of the ſegments of the firſt line, between P and the curve, is to the product of the ſegments of the ſecond line likewiſe between P and the curve, in a conſtant ratio.

Cor. 1. *If the line* AP *cut the curve only in one* 84.
point A; *then if you make* AF $= -\frac{g}{2f}$, *and* CF *perpendicular to it, and* $= \frac{\sqrt{4fh - gg}}{2f}$, *and draw* PC; *then will* PM × Pm × PN : AP × PC² :: f : 1.

For $fx^3 + gx^2 + hx = fx \times \overline{xx + \frac{g}{f}x + \frac{h}{f}}$ $= 0$, (where $k = 0$, becauſe *a* is in the curve), which has only one root x, becauſe AP cuts the curve only in A; and the quantity $xx + \frac{g}{f}x + \frac{h}{f}$, contains two impoſſible roots, viz. $x = \frac{-g}{f} \pm \sqrt{\frac{gg}{4ff} - \frac{h}{f}}$; therefore $\frac{h}{f}$ is greater than $\frac{gg}{4ff}$, or $4fh$ greater than gg. Therefore CF is a real quantity. And PC² = PF² + CF² $= \overline{x + \frac{g}{2f}}^2 +$

Fig. 84. $\frac{4fh-gg}{4ff} = xx + \frac{gx}{f} + \frac{gg}{4ff} + \frac{4fh-gg}{4ff} = xx + \frac{gx}{f} + \frac{h}{f}$. And $fx \times PC^2 = f \times AP \times PC^2 = fx^3 + gx^2 + hx$. Therefore $f \times AP \times PC^2 = PM \times Pm \times PN$; and $AP \times PC^2 : PM \times Pm \times PN :: 1 : f$.

Cor. 2. *And ſince* AF *and* FC *are compoſed of given quantities, the point* C *is fixed; whatever be the angle* APM.

PROP. XVI.

A line of any order may have as many aſſymptotes, as it has dimenſions; and not more.

For let the curve be of the nth order of lines, ſo that its equation is $y^n + \overline{ax+b} \times y^{n-1}$ &c. $= 0$. This equation will have n roots, and the root being extracted will appear in this form, $y = Ax^n + Bx^{n-1}$ &c. Now if by increaſing x ad infinitum, if y be a real quantity the curve will have an infinite leg, and conſequently may have an aſſymptote. And as there are n ſuch roots there may be as many branches and aſſymptotes, for the ſame reaſon; but there can be no more, as there are no more roots of the equation.

PROP. XVII.

The properties of curves of a ſuperior order, agree likewiſe with thoſe of all inferior orders.

An equation of a curve of the third order is denoted thus, $y^3 + \overline{ax+b} \times yy + \overline{cxx+dx+e} \times y + fx^3 + gx^2 + hx + k = 0$. If f, g, h, k, be ſuppoſed $= 0$, then the equation will become $y^2 + \overline{ax + b} \times y + cxx + dx + e = 0$, which is one of a lower order. Now the properties of the curve belonging the firſt equation, muſt hold good of the curve belonging the ſecond equation, with all the quantities that remain; and therefore its properties are included in the former. For all the difference is, that ſome line or lines become 0 and vaniſh, and others become infinite, ſome coincide, others become equal; likewiſe ſome points coincide, and others are removed to an infinite diſtance; yet under theſe circumſtances, the general properties ſtill hold good with the remaining quantities; ſo that whatever is demonſtrated generally of any order, holds true of the inferior orders. And on the contrary, there is hardly any property of the inferior orders, but there is ſome ſimilar to it, in the ſuperior ones.

For as in the conic ſections, if two parallel lines are drawn terminating at the ſection; the right line that biſſects theſe will biſſect all other parallels thereto; and is therefore called the *Diameter* of the figure, and the biſſected lines, *Ordinates*; the interſection thereof with the curve, the *vertex*; and the interſection of all the diameters, the *center*; and that diameter, the *Axis*, which is perpendicular to the ordinates. So likewiſe in higher curves, if two parallel lines are drawn, cutting the curve in a proper number of points; the right line that cuts theſe parallels ſo, that the ſum of the parts on one ſide the line, to the curve; be equal to the ſum of the parts on the other ſide, it will cut all other parallels in the ſame manner, which cut the curve in as many points; then theſe parts may be called *Ordinates*; and the line ſo cutting them, the *Diameter*; the interſection of the diameter and curve, the *Vertex*; the interſection of two diameters, the *Center*; the diameter perpendicular to the ordinates, Fig. ordinates, if there be any, the *Axis.* And when all the diameters concur in one point, that's the *general Center.*

Again, the conic hyperbola being a line of the ſecond gender, has two aſſymptotes; ſo likewiſe that of the third gender may have three; that of the fourth four, and ſo on; and they can have no more. And as the parts of any right line, between the curve and its aſſymptotes are equal; ſo likewiſe in curves of the third ſort of lines, if any line be drawn cutting the curve and its aſſymptotes in three points; the ſum of the two parts of it, drawn the ſame way from the aſſymptotes to the curve; will be equal to the third part drawn the contrary way from the third aſſymptote to the curve. And ſo of higher curves.

Alſo in the conic ſections which are not parabolical. As the ſquare of the ordinate or the rectangle of the parts of it on each ſide the diameter, to the rectangle of the parts of the diameter, terminating at the vertexes; as the latus rectum, to the diſtance of the vertices, or tranſverſe diameter. So in non-parabolic curves of the ſecond gender, the ſolid under the three ordinates, is to the ſolid under the three abſciſſæ or the diſtances to the three vertices; in a certain given ratio. In which ratio, if you take three lines proportional to the three diameters, each to each; then theſe three lines may be called each of them the *Latus Rectum*, and theſe diameters the *Tranſverſe Diameters.* And in the common parabola which has but one vertex for one diameter, the rectangle of the ordinates is equal to the rectangle of the abſciſſa and latus rectum. So in curves of the ſecond kind which have but two vertexes for the ſame diameter. The ſolid under the three ordinates, is equal to the ſolid under the two abſciſſæ,

ſciſſæ, and a given right line, which therefore Fig. may be called the *Latus Rectum.*

Laſtly, ſince in the conic ſections, where two parallel lines terminating at the curve both ways; are cut by two other parallels likewiſe terminating at the curve; make the rectangle of the parts of one of the firſt, to the rectangle of the parts of one of the ſecond lines, paſſing thro' the ſame point of diviſion: as the rectangle of the parts of the ſecond of the former, to the rectangle of the parts of the ſecond of the latter two, paſſing alſo thro' the common point of their diviſion. So when four ſuch lines are drawn in a curve of the ſecond kind, and each meeting it in three points. The ſolid under the parts of the firſt line, will be to that under the parts of the third; as the ſolid under the parts of the ſecond line, to that under the parts of the fourth.

The legs of curves of the ſecond and higher genders, juſt as thoſe of the firſt, running on infinitely; will be either of the *hyperbolic kind*, having ſome aſſymptote; or of the *parabolic kind*, having no aſſymptote. Theſe things appear plainly from the foregoing propoſitions; and therefore it is ſufficiently confirmed, that the properties of curves of a ſuperior order include thoſe of an inferior order.

PROP. XVIII. *Prob.*

To inveſtigate the equation of a curve AMK, *having given the relation of the abſciſſas to one another, and alſo the relation of the ordinates to one another.* 85.

Let the abſciſſa AD, and its ordinate DL be given; and let AD $= a$, DL $= b$, any other abſciſſa AP $= x$, its ordinate PM $= y$. And let x be

Fig. 85. be denoted by a multiplied or divided by the variable quantity v, or ſome power or powers thereof. And likewiſe let y be denoted by b multiplied or divided by ſome powers of v, according to the nature of the figure. Then expunge v and you'll have the equation of the curve.

Ex. 1.

Let AP $= va = x$, and PM $= vvb = y$; then $\frac{y}{b} = vv = \frac{xx}{aa}$, whence $\frac{aa}{b} y = xx$; and the curve is a parabola convex towards AD.

Ex. 2.

Suppoſe AP $= va = x$, and PM $= \frac{b}{v} = y$; then $v = \frac{x}{a} = \frac{b}{y}$; and $xy = ab$; an equation to an hyperbola between the aſſymptotes.

Ex. 3.

Let AP $= va = x$, and PM $= y = \sqrt{vab - vvbb}$; then $v = \frac{x}{a}$, and $y = \sqrt{bx - \frac{bbxx}{aa}}$; an equation to the ellipſis.

Ex. 4.

Let AP $= x = v^m a$, PM $= y = v^n b$. Then $v = \frac{x^{\frac{1}{m}}}{a^{\frac{1}{m}}} = \frac{y^{\frac{1}{n}}}{b^{\frac{1}{n}}}$, whence $b^m x^n = a^n y^m$; a figure of the parabolic kind.

Ex.

Fig. 85.

Ex. 5.

Suppoſe AP $= x = va + v^2a + v^3a$, and PM $= y = vb + v^2b + v^3b$; then is $v + v^2 + v^3 = \frac{x}{a} = \frac{y}{b}$; and $ay = bx$, a right line.

Ex. 6.

Let AP $= x = va$, PM $= y = \frac{b^v}{r^v - 1}$. Then $v = \frac{x}{a}$, and conſequently $y = \frac{b^{\frac{x}{a}}}{r^{\frac{x}{a}} - 1}$; whence $y^a = \frac{b^x}{r^x - a}$; and if $r = 1$, then $y^a = b^x$, and this is the log. curve.

PROP. XIX. *Prob.*

To find the radius of curvature in the vertex of a curve, where the abſciſſa is perpendicular to it.

Let R be the radius of curvature, x the abſciſſa of the curve, y its ordinate. Then inſtead of y in the equation of the curve, put $\sqrt{2Rx}$; and throwing out ſuch quantities as are infinitely leſs than others, and then reducing the equation, R will be had.

For let AGM be any curve, AG an infinitely 86.
ſmall part of the curve, AF its abſciſſa, GF its ordinate perp. thereto, C the center of the circle which paſſes thro' A and G, then CA is the radius of the circle. And ſince the part AG is common to both the curve and the circle, CA will be the radius of curvature required. Put R $=$ CA, AF or BG $= x$, GF or BA $= y$, then by the nature

Fig. 86. ture of the circle $2Rx - xx = yy$, but xx being infinitely leſs than $2Rx$, we ſhall have $yy = 2Rx$, and $y = \sqrt{2Rx}$. Therefore if $\sqrt{2Rx}$ be put for y in the equation of the curve, you'll have R required.

Ex. 1.

Let the curve be an ellipſis, $2t$ the tranſverſe, $2c$ the conjugate, l the latus rectum. Then $2tx - xx : yy :: 2t : l$, and $2tyy = 2tlx - lxx$; that is, (by ſubſtitution) $2t \times 2Rx = 2tlx - lxx =$ (becauſe lxx is infinitely leſs than $2tlx$) $2tlx$; whence $2R = l$, and $R = \frac{1}{2} l$, or half the latus rectum.

Ex. 2.

Let AM be the ciſſoid, whoſe equation (by Cor. 2. Pr. 2. Sect. II. B. I.) is $y^3 = 2axx - yxx$. For y put $\sqrt{2Rx}$; then $2Rx \sqrt{2Rx} = 2axx - yxx =$ (becauſe $2axx$ is infinitely greater than yxx) $2axx$. And ſquaring, $8R^3x^3 = 4aax^4$. And $2R^3 = aax = 0$, becauſe x is infinitely ſmall. Therefore the radius of curvature in the vertex is 0.

Ex. 3

3. Let the curve be the conchoid, whoſe equation (by Prop. IV. B. I. Sect. I.) is $TL \times AL = PL \sqrt{TF^2 - AH^2}$; that is, $\overline{TF - FL} \times AL = \overline{PF - FL} \times \sqrt{TF^2 - \overline{TF - FL}^2}$; that is, putting $TF = a$, $TP = b$, $FP = d$, $FL = x$, $AL = y$, the equation of the curve is $\overline{a - x} \times y = \overline{d - x} \times \sqrt{2ax - xx}$. And in the vertex, throwing out the infinitely ſmall quantities, $ay = d \sqrt{2ax}$, and $aayy = 2ddax$; that is, $aa \times 2Rx = 2ddax$. And $aR = dd$, whence $R = \frac{dd}{a}$, the radius of curvature in the vertex.

Cor.

Fig. 87. Cor. *If the ordinates make not a right angle with the diameter, but any oblique angle* GFH, *put* $s =$ *ſine of the angle* GFH, AF $= x$, FG $= y$, R $=$ *radius of curvature; then inſtead of* y *in the equation of the curve, put* $\sqrt{2Rsx}$, *and rejecting the infinitely ſmall quantities as before, and reducing the equation,* R *will be found.*

For let AFH be the diameter of the curve, AC the radius of curvature, ABD a tangent at A, GB parallel to the abſciſſa AF, GD perp. to AD; and let GD $= v$. Then if the arch AG be ſuppoſed infinitely ſmall, AG $=$ GF, and the part AG of the curve will coincide with the circle. But by the nature of the circle, $AG^2 = DG \times AL$; that is, $yy = 2Rv$. But in the infinitely ſmall triangle DGB, DG (v) : GB or AF (x) : : s.DBG or GFH (s) : radius (1); therefore $v = sx$; whence $yy = 2Rv = 2Rsx$, and $y = \sqrt{2Rsx}$.

Example.

88. Let DGA be a parabola, HAF a diameter, FG an ordinare, l the latus rectum, of the diameter HF; then if AF $= x$, FG $= y$, $s = s$. angle GFH, we ſhall have $lx = yy = 2Rsx$, by the rule, whence $R = \frac{l}{2s}$. This may be denoted otherwiſe, for $1 : s :: AT : AP$, and $s = \frac{AP}{AT} = \frac{AP}{\sqrt{TP^2 + PA^2}} = \frac{AP}{\sqrt{4DP^2 + PA^2}}$, and if $a =$ latus rectum of the axis TP, $s = \frac{AP}{\sqrt{4DP^2 + a \times DP}}$ $= \sqrt{\frac{DP \times a}{4DP^2 + a \times DP}} = \sqrt{\frac{a}{a + 4DP}} = \sqrt{\frac{\frac{1}{4}a}{\frac{1}{4}a + DP}}$. That is, if S be the focus, s $=$

Fig. 88. $= \sqrt{\frac{DS}{SA}}$. Whence the radius of curvature in A $= \frac{1}{2} l \times \sqrt{\frac{SA}{DS}} = \frac{1}{2} l \sqrt{\frac{4SA}{4DS}} = \frac{1}{2} l \sqrt{\frac{l}{a}}$, by the nature of the curve.

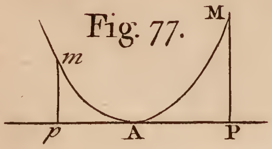
And if P falls upon S, then PA $= \frac{1}{2} a$, and $\frac{1}{2} l = a$, whence R $= a\sqrt{2}$.

PROP. XX.

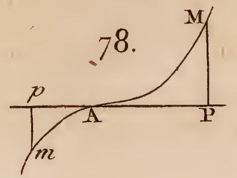
The ſhadow of any given curve line, projected from a lucid point, upon an infinite plane, will be a curve of the ſame gender, with the given one.

For the ſhadow of any point will be in the right line drawn thro' the light and that point. And any right line in the given curve is projected into a right line upon the plane. For imagine a plane drawn thro' the light, and thro' a right line in the given curve, and that this plane is extended to the plane of the ſhadow, it will cut this plane in a right line, which will be the ſhadow of the given line. Therefore right lines that interſect one another will be projected into lines or ſhadows that interſect one another. If any line cuts the curve, its ſhadow will cut the ſhadow of the curve. If any line touches the curve, the ſhadows of the line and curve will touch one another. If one part of the curve interſect another, their ſhadows will alſo interſect. So that all points that are coincident in the figure, will be coincident in the ſhadow. And all points at a diſtance in the figure, will be diſtant from one another in the ſhadow. Now let a right line be drawn in the figure to cut it in as many points, as is the dimenſion of the figure; and it cannot cut it in more. Then the ſhadow of this line will alſo be a right line, and will

Fig. 77.



78.



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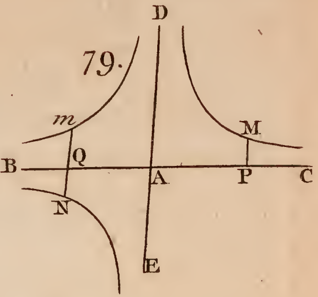

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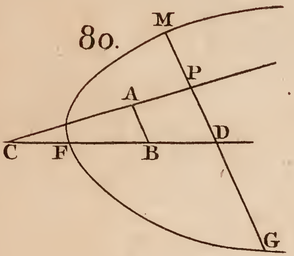

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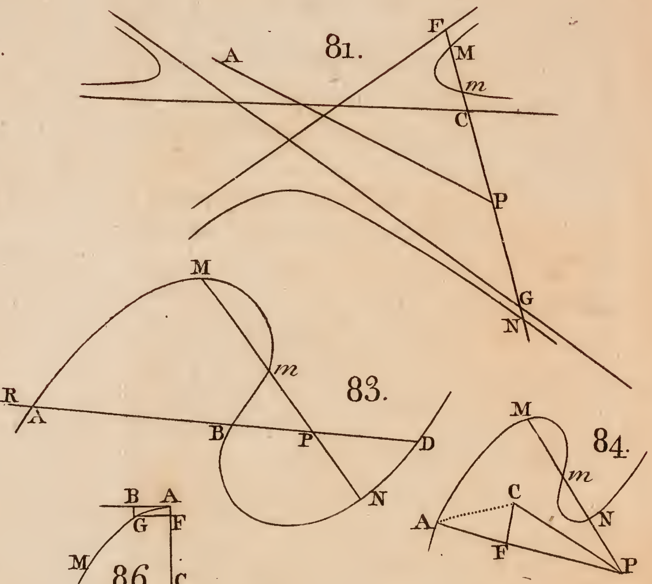

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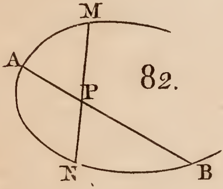

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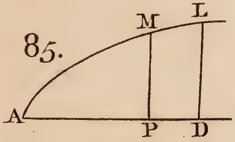

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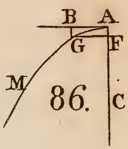
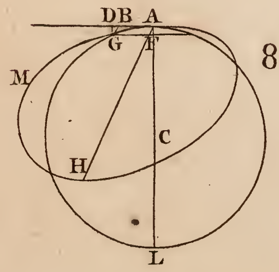
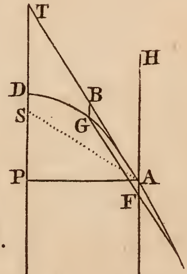


87.



88.



Pl.VII. *pa.92.*

will cut the ſhadow of the curve in as many points, Fig. and no more. Therefore this ſhadow is a curve of the ſame dimenſion, or of the ſame gender with that given.

Cor. 1. *Hence the ſhadow of the circle, or any conic ſection, will be a conic ſection. The ſhadow of lines of the third order, will be lines of the third order, and ſo on ad infinitum.*

Cor. 2. *If any part of the curve be higher (or further from the plane) than the light is; that part of the curve can caſt no ſhadow upon the plane; and the ſhadow of the curve will run on infinitely towards that hand. And therefore whatever there is in that part of the curve above the light, will be quite loſt in the ſhadow, and no where be found.*

SECT. III.

Some fluxionary Propoſitions neceſſary for inveſtigating the nature of Curves.

Fig. HAVING handled ſeveral things of this nature in my book of Fluxions, I intend not to repeat them here; but ſhall only take notice of ſuch as have been omitted there, or but ſlightly mentioned. And this I the rather do, becauſe many properties of curves of high genders are far more eaſily inveſtigated by the method of fluxions, than by common algebra.

PROP. I.

89. *Let* AM *be any curve,* AP *the abſciſſa,* PM *an ordinate perp. to it,* MC *the radius of curvature in* M, CF *parallel to* AP *meeting the ordinate* MPF *in* F. *Then the ſecond fluxion of the ordinate, the fluxion of the curve, and the line* MT, *are continually proportional; ſuppoſing the fluxion of the axis to be conſtant.*

Let M, m, r be three points in the curve, $\mathrm{M}o$ and ms parallel to AP. Produce $\mathrm{M}m$ to n, then ſince $\mathrm{M}o = mr$, therefore $\mathrm{M}m = mn$; then $rn =$ fluxion of om, or the ſecond fluxion of the ordinate. Draw $m\mathrm{F}$, then the triangles mrn, $\mathrm{MF}m$ are ſimilar, for $< nrm =$ alternate angle $m\mathrm{MF}$, and $mnr = \mathrm{M}mo = \mathrm{M}m\mathrm{F}$; and therefore $nr : rm :: \mathrm{M}m : \mathrm{MF}$, but $rm = \mathrm{M}m$, therefore $nr : \mathrm{M}m :: \mathrm{M}m : \mathrm{MF}$.

Cor. 1. *Hence, the fluxion of the axis* Mo : *to fluxion of the curve* Mm : : MF : MC, *the radius of curvature in* M. Fig. 89.

Cor. 2. *Hence the radius of curvature in any curve is eaſily found; by firſt finding* MF *by this Prop. And then* MC *by Cor.* 1.

Example.

Let AP $= x$, PM $= y$, AM $= z$, and let the equation of the curve be $2ax + xx = zz$, this put into fluxions, is $2a\dot{x} + 2x\dot{x} = 2z\dot{z}$, or $a\dot{x} + x\dot{x} = z\dot{z}$; then $\dot{z} = \frac{a+x}{z}\dot{x}$, and $\dot{x}^2 + \dot{y}^2 = \dot{z}^2 = \frac{\overline{a+x}^2}{zz}\dot{x}^2$, and $\dot{y}^2 = \frac{\overline{a+x}^2}{zz}\dot{x}^2 - \dot{x}^2 = \frac{\overline{a+x}^2}{2ax+xx}\dot{x}^2 - \frac{2ax+xx}{2ax+xx}\dot{x}^2 = \frac{aa\dot{x}^2}{2ax+xx}$, and $\dot{y} = \frac{a\dot{x}}{\sqrt{2ax+xx}}$.

Then $\ddot{y} = -\frac{a\dot{x} \times \overline{a\dot{x}+x\dot{x}}}{\overline{2ax+xx}^{\frac{3}{2}}}$. Then

$$\frac{a\dot{x} \times \overline{a+x} \times \dot{x}}{\overline{2ax+xx}^{\frac{3}{2}}}\,(\ddot{y}) : \frac{a+x}{z}\dot{x}\,(\dot{z}) :: \frac{a+x}{z}\dot{x}\,(\dot{z}) : \frac{\overline{a+x}^2 \times \dot{x}^2}{zz} \times \frac{\overline{2ax+xx}^{\frac{3}{2}}}{a\dot{x}\dot{x} \times \overline{a+x}} = \text{MF} = \frac{a+x}{a}z.$$

Then $\dot{x} : \dot{z}\left(\frac{a+x}{z}\dot{x}\right) :: \text{MF}\left(\frac{a+x}{a}z\right) : \text{MC} = \frac{\overline{a+x}^2}{a}$, the radius of curvature.

And in the vertex where x is 0, the radius of curvature $= a$. This curve is called the *Catenary*.

Fig.

PROP. II.

90. *If* PF, QD, RC *be three equidiſtant ordinates in any curve and infinitely near; and* nDb *be a tangent at* D. *Then the ſubtenſe* nF *or* bC, *will be equal to half the ſecond fluxion of the ordinate.*

For draw FDa; then ſince $nD = Db$, and the ſubtenſes nF, bC parallel, and the curvature FDC the ſame throughout; therefore $nF = bC$. Again, in the triangles nDF, bDa, ſince $nD = Db$, and all the angles equal; therefore $nF = ab = bC$. Therefore $2bC = aC$, and $bC = \frac{1}{2}aC$.
Let FL and DO be parallel to the axis PR, and and let $PF = y$, then $LD = \dot{y} = Oa$, and $Oa - OC$ or $aC = \ddot{y}$, and bC or $nF = \frac{1}{2}aC = \frac{1}{2}\ddot{y}$.

Cor. *If* PF, QD, RC *be three equidiſtant ordinates, and* FD *be drawn, cutting* RC *in* a. *Then* $Ca =$ *ſecond fluxion of the ordinate.*
For $aC = 2bC = \ddot{y}$, by the demonſtration.

PROP. III.

91. *In any curve whoſe ordinates are drawn from a fixed point* C. *Let* CM, Cm, Cn *be three ordinates infinitely near, ſo that the perpendiculars* Mo, mp *may be equal. And if* Mm *be drawn interſecting* Cn *in* s. *And if* $Cm = y$, $mp = \dot{x}$, $mn = \dot{z}$. *Then the line* $ns = \ddot{y} - \frac{\dot{z}\dot{z}}{y}$.

For draw mr between s and C, making the angle $smr = mCn$; then the triangles Cmn, mrs will be ſimilar, for the angles at m and C are equal by conſtruction, and msC or $msr = mnC$; therefore

Cm

Fig. 91. Cm (y) : mr ($\dot{z}$) : : mr ($\dot{z}$) : $rs = \frac{\dot{z}\dot{z}}{y}$. Draw mq parallel to Mo. Then the angle $pmq = m$C$n = smr$, whence the angle $rmp = smq = m$Mo, and M$o = mp$. Therefore the triangles Mmo, mrp are ſimilar and equal, whence $pr = mo$. Therefore $\ddot{y} = np - mo = np - rp = rn$. But $sn = rn - rs$; therefore $sn = \ddot{y} - \frac{\dot{z}\dot{z}}{y}$.

Cor. *In any curve whoſe ordinates are referred to a fixed point; if* CM, Cm, *be two ordinates, and* mt *a tangent at* m; *then the ſubtence* Mt $= \frac{y\ddot{y} - \dot{z}\dot{z}}{2y}$.

For Mt is half of $sn = \frac{1}{2}\ddot{y} - \frac{\dot{z}\dot{z}}{2y}$.

PROP. IV.

If $\dot{y}$, $\ddot{y}$, $\dot{\ddot{y}}$, *&c. denote the firſt, ſecond, third, &c. moment, increment, or fluxion of* y; *and if* P, Q, R, S, T, *&c. be ſeveral equidiſtant ordinates in a curve, being infinitely near one another. And let the firſt* P *be called* y. *Then will* 92.

$$\dot{y} = \dot{y}$$
$$\dot{q} = \dot{y} + \ddot{y}$$
$$\dot{r} = \dot{y} + 2\ddot{y} + \dot{\ddot{y}}$$
$$\dot{s} = \dot{y} + 3\ddot{y} + 3\dot{\ddot{y}} + \ddot{\ddot{y}}$$
&c.

For by the nature of curvature $\dot{q} = \dot{p} + \ddot{p}$; that is, $\dot{q} = \dot{y} + \ddot{y}$, $\dot{r} = \dot{q} + \ddot{q}$, $\dot{s} = \dot{r} + \ddot{r}$, &c. Therefore taking the fluxions of theſe equations, Fig. 92. $\ddot{q} = \ddot{y} + \dddot{y}$, $\ddot{r} = \ddot{q} + \dddot{q}$, $\ddot{s} = \ddot{r} + \dddot{r}$, $\dddot{q} = \dddot{y} + \ddddot{y}$, &c. Whence $\dot{r} = \dot{q} + \ddot{q} = \dot{y} + \ddot{y} + \ddot{y} + \dddot{y} = \dot{y} + 2\ddot{y} + \dddot{y}$, $\dot{s} = \dot{r} + \ddot{r} = \dot{q} + \ddot{q} + \ddot{q} + \dddot{q} = \dot{q} + 2\ddot{q} + \dddot{q} = \dot{y} + \ddot{y} + 2\ddot{y} + 2\dddot{y} + \dddot{y} + \ddddot{y} = \dot{y} + 3\ddot{y} + 3\dddot{y} + \ddddot{y}$, &c.

Cor. *Hence the fluxion of the nth ordinate* P, Q, R, S, *&c. is* $= \dot{y} + \overline{n-1} \,.\, \ddot{y} + \frac{\overline{n-1} \,.\, \overline{n-2}}{2} \dddot{y} + \frac{\overline{n-1} \,.\, \overline{n-2} \,.\, \overline{n-3}}{2 \,.\, 3} \ddddot{y}$ + *&c.*

PROP. V

93. 94. *Let* M *be the point of contrary flexion in any curve* AM; *and if curve* AM $= z$, *ordinate* PM *or* CM $= y$, M$o = \dot{x}$. *Then in curves referred to an axis,* $\ddot{y} = 0$, *and in ſpirals* $y\ddot{y} = \dot{z}^2$, *ſuppoſing* $\dot{x}$ *to be given.*

93. For at M the point of contrary flexion, three points n, M, m, infinitely near one another, lie in a right line. Therefore in curves referred to an axis, if Mo or $\dot{x}$ be given, om or $\dot{y}$ will be given, conſequently $\ddot{y} = 0$.

94. And in ſpirals let CT be perp. to the tangent MT; then ſince TM is a tangent at all the points n, M, m. Therefore at M, CT is a conſtant quantity; let CT $= t$, then by ſimilar triangles $\dot{z} : \dot{x} :: y :$ CT $= \frac{y\dot{x}}{\dot{z}} = t$, and $y\dot{x} = t\dot{z}$, in fluxions, $\dot{y}\dot{x} = t\ddot{z}$, $\dot{x}$ and t being conſtant quantities. Therefore $\dot{y}\dot{x} = \frac{y\dot{x}}{\dot{z}} \ddot{z}$, and $\dot{y}\dot{x}\dot{z} = y\dot{x}\ddot{z}$. But

$\dot z^2 = \dot x^2 + \dot y^2$; therefore $\dot z\ddot z = \dot y\ddot y$, and $\ddot z = \frac{\dot y\ddot y}{\dot z}$. Therefore $y\dot x\dot z = y\dot x \times \frac{\dot y\ddot y}{\dot z}$, and $\dot z = \frac{y\ddot y}{\dot z}$, and $\dot z\dot z = y\ddot y$. Fig. 94.

Cor. *Hence to find the point of contrary flexion, let $\dot x$ be given, and put $\ddot y = 0$, in curves referred to an axis; or $\ddot y = \frac{z^2}{y}$ in ſpirals.*

Ex. 1.

Let the equation of the curve $b^3 \times \overline{y - a}^2 = \overline{x - a}^5$; then $\overline{y - a}^2 = \frac{\overline{x - a}^5}{b^3}$, and $y - a = \frac{\overline{x - a}^{\frac{5}{2}}}{b\sqrt{b}}$, and $y = a + \frac{\overline{x - a}^{\frac{5}{2}}}{b\sqrt{b}}$, whence $\dot y = \frac{5}{2} \times \frac{\overline{x - a}^{\frac{3}{2}}}{b\sqrt{b}} \times \dot x$, and $\ddot y = \frac{15}{4b\sqrt{b}} \times \overline{x - a}^{\frac{1}{2}} \times \dot x\dot x = 0$; therefore $\overline{x - a}^{\frac{1}{2}} = 0$, and $x - a = 0$. Therefore $x = a$.

Ex. 2.

Let ABM be a ſort of ſpiral, RFB its generating circle, radius CD $= r$, CM $= y$, arch RD $= v$. And let the nature of it be denoted by the equation $vyy = a^3$. 95.

The ſectors CD*d*, and CM*o* are ſimilar, whence CM (y) : M*o* $(\dot x)$: : CD (r) : D*d* $= \frac{r\dot x}{y} = \dot v$, and by the equation of the curve $yy\dot v + 2vy\dot y = 0$, and $\dot v = \frac{-2v\dot y}{y} = \frac{r\dot x}{y}$, and $\dot y = \frac{-r\dot x}{2v} =$ (becauſe $v = \frac{a^3}{yy}$) $\frac{-ryy\dot x}{2a^3}$, whence $\ddot y = \frac{-ry\dot y\dot x}{a^3}$, and

Fig. 95. $y\ddot{y} = \frac{-ry^2\dot{y}\dot{x}}{a^3} = \frac{r^2y^4\dot{x}^2}{2a^6}$. Alſo $\dot{z}^2 = \dot{x}^2 + \dot{y}^2 = \dot{x}^2 + \frac{rry^4\dot{x}^2}{4a^6}$. Whence $\frac{r^2y^4\dot{x}^2}{2a^6} = \dot{x}^2 + \frac{rry^4\dot{x}^2}{4a^6}$, and $2r^2y^4 = 4a^6 + r^2y^4$. And $r^2y^4 = 4a^6$, and $ry^2 = 2a^3$; whence $y = \sqrt{\frac{2a^3}{r}}$. And if $r = a$, then $y = r\sqrt{2}$.

PROP. VI.

If x, y, z, &c. repreſent any right lines; and in any curve, if $x \times y \times z$, &c. $=$ a given quantity; then $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z}$ &c. $= 0$.

For let $xyz = a$, then the fluxion will be $zy\dot{x} + xz\dot{y} + xy\dot{z} = 0$; divide by the given equation; then $\frac{zy\dot{x}}{zyx} + \frac{xz\dot{y}}{xzy} + \frac{xy\dot{z}}{xyz} = \frac{0}{xyz}$; that is, $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z}$, &c. $= 0$.

Cor. *If xy^mz^n &c. $= a$, a given quantity; then $\frac{\dot{x}}{x} + \frac{m\dot{y}}{y} + \frac{n\dot{z}}{z}$ &c. $= 0$.*

PROP.

PROP. VII.

If x, y, z, v, u, w, *&c. repreſent any right lines, and in any curve if* $\frac{xyz\ \&c.}{vuw\ \&c.}$ = *any given quantity; then* $\frac{\dot{x}}{x}+\frac{\dot{y}}{y}+\frac{\dot{z}}{z}$ *&c.* $-\frac{\dot{v}}{v}-\frac{\dot{u}}{u}-\frac{\dot{w}}{w}$ *&c.* $=0$. 95.

Let $\frac{xyz}{vuw}=a$, then $\frac{\dot{x}yz}{vuw}+\frac{x\dot{y}z}{vuw}+\frac{xy\dot{z}}{vuw}-\frac{xyz\dot{v}}{v^2uw}-\frac{xyz\dot{u}}{vu^2w}-\frac{xyz\dot{w}}{vuw^2}$ &c. $=0$. And dividing all by $\frac{xyx}{vuw}=a$, we ſhall have $\frac{\dot{x}}{x}+\frac{\dot{y}}{y}+\frac{\dot{z}}{z}-\frac{\dot{v}}{v}-\frac{\dot{u}}{u}-\frac{\dot{w}}{w}$ &c. $=0$.

Cor. *If* $\frac{xy^m\ \&c.}{vu^n\ \&c.}$, *then* $\frac{\dot{x}}{x}+\frac{m\dot{y}}{y}$ *&c.* $-\frac{\dot{v}}{v}-\frac{n\dot{u}}{u}$ *&c.* $=0$.

Fig.

PROP. VIII.

96. *If from a given point* P *the right lines* PA, P*a*, *be drawn, cutting any curve in as many points* A, B, C, *and a, b, c. as it has dimenſions. And the tangents* AK, BL, CM; *and ak, bl, cm, be drawn, cutting any line* PE, *in the points* K, L, M, *and k, l, m. Then will* $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM}$ *&c.* = $\frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm}$ *&c. wherever the points* A, B, C, *and a, b, c, fall.*

Let AC, *ac* be ſuppoſed to move parallel to themſelves, and then the point of interſection P will move along the right line PE. Then ſince (Prop. XV. Sect. II.) PA × PB × PC : : P*a* × P*b* × P*c* : : *f* : 1, a given ratio. Let AP = *x*, PB = *y*, PC = *z*, and PA = *v*, P*b* = *u*, P*c* = *w*; then $xyz = fvuw$, or $\frac{xyz}{vuw} = f$. Therefore (Prop. VII.) $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} - \frac{\dot{v}}{v} - \frac{\dot{u}}{u} - \frac{\dot{w}}{w} = 0$, and $\frac{\dot{x}}{x} + \frac{\dot{y}}{y} + \frac{\dot{z}}{z} = \frac{\dot{v}}{v} + \frac{\dot{u}}{u} + \frac{\dot{w}}{w}$. But ſince AP is moved with a parallel motion, it is evident the fluxion of AP is to the fluxion of EP, as AP to the ſubtangent PK; whence, putting EP = *s*, we have $\frac{\dot{x}}{x} = \frac{\dot{s}}{PK}$, and for the ſame reaſon $\frac{\dot{y}}{y} = \frac{\dot{s}}{PL}$, and $\frac{\dot{z}}{z} = \frac{\dot{s}}{PM}$; alſo $\frac{\dot{v}}{v} = \frac{\dot{s}}{Pk}$, $\frac{\dot{u}}{u} = \frac{\dot{s}}{Pl}$, $\frac{\dot{w}}{w} = \frac{\dot{s}}{Pm}$. Whence $\frac{\dot{s}}{PK} + \frac{\dot{s}}{PL} + \frac{\dot{s}}{PM} = \frac{\dot{s}}{Pk} + \frac{\dot{s}}{Pl}$ +

$+ \frac{s}{Pm}$; and $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} = \frac{1}{Pk} + \frac{1}{Pl} + \frac{1}{Pm}$. Fig. 96.

But if any of the points K, L, M, &c. fall on the contrary ſide of P, then the correſpondent lines PK, PL, PM, &c. muſt be made negative.

Cor. 1. *If any line* PC *cut the curve in the points* A, B, C, *and tangents drawn; then will* $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM}$ *&c. be always equal to an invariable quantity.*

Cor. 2. *And hence if a right line* PE *cut the curve in* D, E, I, *&c. as many points as it has dimenſions; then will* $\frac{1}{PK} + \frac{1}{PL} + \frac{1}{PM} = \frac{1}{PD} + \frac{1}{PI} + \frac{1}{PE}$.

For in this caſe *k*, *l*, *m*, coincide with D, I, E.

Cor. 3. *If any of the points* K, L, M, *&c. fall on the contrary ſide of* P; *you muſt take* $-\frac{1}{PK}, -\frac{1}{PL}, -\frac{1}{PM}$, *&c.*

For lying the contrary way, they muſt have contrary ſigns.

PROP.

Fig.

PROP. IX.

97. *If the line* AB *be drawn thro' a given point* P *parallel to the aſſymptote* CM; *and the tangents* AK, BL *drawn, cutting any line* DE (*drawn alſo thro'* P, *and cutting the curve in three points* D, I, E; *and the aſſymptote in* M). *Then* $\frac{1}{PM} = \frac{1}{PI} + \frac{1}{PE} - \frac{1}{PD} - \frac{1}{PK} - \frac{1}{PL}$.

For (by Prop. VII.) $\frac{1}{PM} + \frac{1}{PK} + \frac{1}{PL} = \frac{1}{PI} + \frac{1}{PE} - \frac{1}{PD}$, becauſe D lies the contrary way from P, and AB meets the curve and aſſymptote CM at an infinite diſtance where they touch. Then by tranſpoſing, $\frac{1}{PM} = \frac{1}{PI} + \frac{1}{PE} - \frac{1}{PD} - \frac{1}{PK} - \frac{1}{PL}$.

Cor. *Hence the diſtance of the aſſymptote may be found, by finding the point* M. *And if* $\frac{1}{PM}$ *be* $= 0$, *then the aſſymptote is at an infinite diſtance, and the curve will be of the parabolical kind.*

PROP. X.

To find the aſſymptotes of a curve.

98. This Prob. may be ſolved by the laſt Prop. and its Corol. or otherwiſe as follows. Let AP $= x$, PM $= y$, TM a tangent at M, and TP the ſubtangent. Find the value of the ſubtangent $\frac{y\dot{x}}{\dot{y}}$ by the equation of the curve, then putting $x =$ infinity in the value of the ſubtangent, and alſo in

the equation of the curve, and you will get TP and PM, which gives the poſition of the aſſymptote TM, and TP — x, gives the interſection T. Fig. 98.

For the triangles Mrm and TPM are ſimilar, and rm $(\dot{y})$: Mr $(\dot{x})$: : MP (y) : TP $= \frac{y\dot{x}}{\dot{y}}$. And ſuppoſing M at an infinite diſtance x will be infinite, from whence both TP and PM will be had, and conſequently the poſition of TM the aſſymptote.

Ex. 1.

Let the equation of the curve DM be $xyy = a^3$; in fluxions $y^2\dot{x} + 2xy\dot{y} = 0$, and $\dot{x} = \frac{-2xy\dot{y}}{yy} = \frac{-2x\dot{y}}{y}$. Therefore PT $= \frac{y\dot{x}}{\dot{y}} = -2x$. And PM $= y = \sqrt{\frac{a^3}{x}}$. Then making x infinite, PT $= -2x = -$ infinity, and PM $= \sqrt{\frac{a^3}{x}} = 0$; therefore the aſſymptote coincides with the axis TP.

Ex. 2.

Let $aayy = 2abbx + bbxx$. In fluxions $2aay\dot{y} = 2abb\dot{x} + 2bbx\dot{x}$, and $\dot{x} = \frac{aay\dot{y}}{abb + bbx}$; therefore PT $= \frac{y\dot{x}}{\dot{y}} = \frac{aayy}{abb + bbx}$; and $y = \sqrt{\frac{2abbx + bbxx}{aa}}$. Make x infinite, then PT $= \frac{aayy}{abb + bbx} = \frac{aayy}{bbx} = \frac{2abbx + bbxx}{bbx} = \frac{bbxx}{bbx} = x$ = infinity, and PM $= y = \sqrt{\frac{bbxx}{aa}} = \frac{bx}{a} =$ infinity. Therefore PT : PM : : $x : \frac{bx}{a}$: : $a : b$. Al-

ſo

Fig. 98. ſo AT $= \text{TP} - \text{AP} = \frac{aayy}{abb + bbx} - x =$ $\frac{2abbx + bbxx - abbx - bbxx}{abb + bbx} = \frac{abbx}{abb + bbx} =$ (when x is infinite) $\frac{abbx}{bbx} = a$. Therefore the point T is determined, and the poſition of the aſſymptote TM.

Alſo ſince y may be $= -\sqrt{\frac{bbxx}{aa}} = -\frac{bx}{a}$, this gives another aſſymptote on the other ſide of the axis AP, and making the ſame angle.

Ex. 3.

Let the equation of the curve be, $y^3 - aay + axy - x^3 = 0$. Extracting the root by infinite ſeries, then $y = x - \frac{1}{3}a + \frac{aa}{3x}$ &c. The equation of the curve put into fluxions, gives $3y^2\dot{y} - aa\dot{y} + ax\dot{y} + ay\dot{x} - 3x^2\dot{x} = 0$, whence $\dot{x} = \frac{aa - ax - 3yy}{ay - 3xx}\dot{y}$, and PT $= \frac{y\dot{x}}{\dot{y}} = \frac{ax + 3yy - aa}{3xx - ay}y$, and AT $= \frac{ax + 3yy - aa}{3xx - ay}y - x =$ $\frac{2axy + 3y^3 - aay - 3x^3}{3xx - ay} =$ (expunging y^3) $\frac{2axy - aay - 3x^3 + 3aay - 3axy + 3x^3}{3xx - ay} =$ $\frac{2aay - axy}{3xx - ay} =$ (expunging y) $\frac{\overline{-ax + 2aa} \times \overline{x - \frac{1}{3}a}}{3xx - ax \text{ \&c.}}$ $= \frac{-axx}{3xx} = -\frac{1}{3}a$. But PT $= \frac{axy + 3y^3 = aay}{3xx - ay} =$ (expunging y^3) $\frac{axy - aay + 3x^3 + 3aay - 3axy}{3xx - ay} =$

2aay

$\frac{2aay - 2axy + 3x^3}{3xx - ay} = \frac{3x^3 - \overline{2ax - 2aa} \times y}{3xx - a \times y} =$ Fig. 98.

(expunging y) $\frac{3x^3 - 2ax^2 \&c}{3x^2 - ax \&c.} = \frac{3x^3}{3x^2} = x$, whence PT : PM : : $x : y$ or $x - \frac{1}{3}a : : x : x$; therefore PT = PM. And therefore ſince AT is negative, T falls between A and P; and the aſſymptote therefore lies between the curve and abſciſſa.

Ex. 4.

Let the equation of the curve be $y^3 + axy - x^3 = 0$, whoſe root is $y = x - \frac{1}{3}a + \frac{a^3}{81x^2}$ &c. then the equation in fluxions is $3y^2\dot{y} + ax\dot{y} + ay\dot{x} - 3x^2\dot{x} = 0$, whence $\dot{x} = \frac{3yy + ax}{3xx - ay}\dot{y}$, and PT $= \frac{y\dot{x}}{\dot{y}} = \frac{3y^3 + axy}{3xx - ay} =$ (expunging y^3) $\frac{3x^3 - 3axy + axy}{3xx - ay} =$ (expunging y) $\frac{3x^3 - 2ax \times \overline{x - \frac{1}{3}a}}{3xx - a \times \overline{x - \frac{1}{3}a}} = \frac{3x^3 - 2ax^2 \&c.}{3xx - ax \&c.} = \frac{3x^3}{3xx} = x$. Whence PT : PM : : $x : y$ or $x - \frac{1}{3}a$ &c. : : $x : x$. Whence again PT = PM. Alſo TP $- x = \frac{3x^3 - 2axy}{3xx - ay} - x = \frac{-axy}{3xx - ay} =$ (expunging y) $\frac{-ax \times \overline{x - \frac{1}{3}a}}{3xx - a \times \overline{x - \frac{1}{3}a}} = \frac{-axx - \frac{1}{3}aax \&c}{3xx - ax + \frac{1}{3}aa \&c.} = \frac{-axx}{3xx} = -\frac{1}{3}a$; whence the ſame concluſion will follow as in the laſt.

Ex. 5.

Suppoſe the equation of the curve is $y^3 - axy - x^3 = 0$; this in fluxions is $3y^2\dot{y} - ax\dot{y} - ay\dot{x} - 3x^2\dot{x} =$

Fig. 98. $= 0$; and $\dot{x} = \frac{3yy - ax}{3xx + ay}\dot{y}$, and PT $= \frac{y\dot{x}}{\dot{y}} =$ $\frac{3y^3 - axy}{3xx + ay} =$ (expunging y^3) $\frac{3x^3 + 3axy - axy}{3xx + ay} =$ $\frac{3x^3 + 2axy}{3xx + ay}$. And the root of the equation is $y = x + \frac{1}{3}a$ &c. Therefore PT $=$ $\frac{3x^3 + 2ax \times \overline{x + \frac{1}{3}a}}{3xx + a \times \overline{x + \frac{1}{3}a}} = \frac{3x^3 + 2ax^2 \text{ \&c.}}{3x^2 + ax \text{ \&c.}} =$ (when x is infinite) $\frac{3x^3}{3x^2} = x$. Therefore PT : PM :: x : y or $x + \frac{1}{3}a$ :: $x : x$, and PT $=$ PM.

Alſo PT $- x = \frac{3x^3 + 2axy}{3xx + ay} - x = \frac{axy}{3xx + ay}$ $= \frac{ax \times \overline{x + \frac{1}{3}a}}{3xx + a \times \overline{x + \frac{1}{3}a}} = \frac{ax^2 \text{ \&c.}}{3x^2 \text{ \&c.}} = \frac{1}{3}a =$ AT. Therefore ſince AT is affirmative, the curve is concave to the abſciſſa, and the curve lies between the abſciſſa and the aſſymptote.

PROP. XI.

99. *Let* AMN *be any curve,* CN *its aſſymptote; then if the ordinates be parallel to the aſſymptote; they are depreſſed a dimenſion lower in the equation of the curve.*

Let AP $= x$, ordinate PM $= y$, and let the equation of the curve be $y^n + ax^{n-1}$ &c. $= r^n x^n$, and the root $y = rx + B + \frac{C}{x}$ &c. Take s infinitely near M, and draw so parallel to AP, and DM parallel to CN; then by ſimilar triangles, Mo ($\dot{y}$) : so ($\dot{x}$) : : MP (y) : PT $= \frac{y\dot{x}}{\dot{y}}$ the ſubtangent. But

$\dot{y} = r\dot{x} - \frac{C\dot{x}}{xx}$ &c. and therefore PT $= \frac{y\dot{x}}{r\dot{x} - \frac{C\dot{x}}{xx}}$ Fig. 99.

$= \frac{y}{r - \frac{C}{xx}}$; and when x is infinite, the ſubtangent $= \frac{y}{r} =$ CQ, CN the aſſymptote being a tangent at an infinite diſtance. Draw AB parallel to PM, and let CA $= t$, CB $= d$, AB $= b$, CD $= v$, DM $= z$. Then by ſimilar triangles, we have CQ $\left(\frac{y}{r}\right)$: QN (y) : : CA (t) : AB (b), whence $ty = \frac{by}{r}$, and $tr = b$. Alſo AB (b) : BC (d) : : PM (y) : DM (z), whence $y = \frac{bz}{d}$. Alſo AB (b) : AC (t) : : PM (y) : PD $= \frac{ty}{b}$. Then AP $=$ PC $-$ AC; that is, $x = \frac{ty}{b} + v - t =$ (expunging y) $\frac{tz}{d} + v - t$. Then ſubſtituting theſe quantities inſtead of y and x, in the equation of the curve, $\frac{b^n z^n}{d^n} + a \times \overline{\frac{tz}{d} + v - t}\,\Big|^{n-1}$ &c. $= r^v \times \overline{\frac{tz}{d} + v - t}\,\Big|^{n} = \frac{r^n t^n z^n}{d^n}$ &c. But $b = rt$, and $b^n = r^n t^n$; therefore z^n the higheſt power of the ordinate vaniſhes; therefore when the ordinate DM is parallel to the aſſymptote, it loſes one dimenſion in the equation of the curve.

Fig.

PROP. XII.

100. *In any curve referred to an axis, if the ſecond fluxion of the ordinate* PM *be negative; the curve is concave towards the axis* AP. *But if the ſecond fluxion of the ordinate be affirmative, the curve is convex towards the axis; ſuppoſing the fluxion of the axis given.*

Draw the ordinates *pm*, *qn* parallel to PM, at equal but infinitely ſmall diſtances. Produce the infinitely ſmall portion of the curve M*m* to *s*. And draw M*o*, *mr* parallel to the axis AP. Then the fluxion of the axis is $Mo = mr$, and conſequently $mo = sr$, the fluxion of the ordinate. Now $rn - mo$ is the ſecond fluxion of the ordinate; that is, $rn - rs$ is the ſecond fluxion, and this will be negative when *rn* is leſs than *rs*, in which caſe M*mn* is concave towards AP. But if $rn - rs$ is affirmative, then *rn* is greater than *rs*, and then *n* falls above *s*, and therefore the curve is convex towards AP.

Cor. *From hence may be known whether a curve, referred to an axis, is concave or convex towards the axis in any given point. Put the equation of the curve into fluxions, and find the value of the ſecond fluxion of the ordinate, making the fluxion of the axis conſtant. Then if the value of that ſecond fluxion be negative, the curve is concave; if affirmative, 'tis convex to the axis.*

Ex. 1.

Let the equation of the curve be $aayy = 2bbrx + bbxx$, this put into fluxions, and divided by 2, gives $aay\dot{y} = bbr\dot{x} + bbx\dot{x}$, and put in fluxions again, $aa\dot{y}\dot{y} + aay\ddot{y} = bb\dot{x}\dot{x}$, and $aay\ddot{y} = bb\dot{x}\dot{x} - aa\dot{y}\dot{y}$,

Fig. 100.

$aa\dot{y}\dot{y}$, *but* $\dot{y} = \frac{bb\dot{x}}{aay} \times \overline{r+x}$, and $\dot{y}^2 = \frac{b^4\dot{x}^2}{a^4yy} \times \overline{r+x}^2$, whence $aay\ddot{y} = bb\dot{x}\dot{x} - \frac{b^4\dot{x}^2}{aayy} \times \overline{r+x}^2 = \frac{bbaayy - b^4 \times \overline{r+x}^2}{aayy}\dot{x}^2 =$ (expunging yy) $-\frac{b^4rr\dot{x}\dot{x}}{aayy}$; therefore $\ddot{y}$ being negative, the curve is concave to the axis.

Ex. 2.

Let the equation of the curve be $aay + xxy = axx$; then $y = \frac{axx}{aa+xx}$, and $\dot{y} = \frac{2ax\dot{x}}{aa+xx} - \frac{2ax^3\dot{x}}{\overline{aa+xx}^2} = \frac{2a^3x\dot{x} + 2ax^3\dot{x} - 2ax^3\dot{x}}{\overline{aa+xx}^2} = \frac{2a^3x\dot{x}}{\overline{aa+xx}^2}$ and $\ddot{y} = \frac{2a^3\dot{x}\dot{x}}{\overline{aa+xx}^2} - \frac{2a^3x\dot{x} \times 4x\dot{x}}{\overline{aa+xx}^3} = \frac{2a^3\dot{x}\dot{x} \times \overline{aa+xx} - 8a^3xx\dot{x}\dot{x}}{\overline{aa+xx}^3} = \frac{2a^5\dot{x}^2 - 6a^3x^2\dot{x}^2}{\overline{aa+xx}^3} = \frac{2a^3\dot{x}^2}{\overline{aa+xx}^3} \times \overline{aa-3xx}$. Therefore if $3xx$ be greater than aa, the curve is concave towards the axis; but if aa be greater than $3xx$, it is convex.

Ex. 3.

If the equation of the curve be $aax - a^3 = \overline{y-b}^3$. Then $aa\dot{x} = 3\dot{y} \times \overline{y-b}^2$, and again $3\ddot{y} \times \overline{y-b}^2 + 3\dot{y} \times 2\dot{y} \times \overline{y-b} = 0$, and $\ddot{y} \times \overline{y-b} + 2\dot{y}\dot{y} = 0$, and $\ddot{y} = -\frac{2\dot{y}\dot{y}}{y-b}$. But $\dot{y} = \frac{aa\dot{x}}{3 \times \overline{y-b}^2}$; therefore $\ddot{y} = -\frac{2a^4\dot{x}\dot{x}}{9 \times \overline{y-b}^3}$; conſequently if y be greater than b, the curve is concave towards the

Fig. 100. axis; but if b be greater than y, the curve is convex towards the axis.

PROP. XIII.

91. *In a curve where the ordinates meet in a fixed point, let the ordinate* CM *or* C$m = y$, *perp.* M$o = \dot{x}$, M$m = \dot{z}$; *then ſuppoſing $\dot{x}$ to be conſtant, if $y\ddot{y} - \dot{z}\dot{z}$ be negative, the curve is concave towards the center; but if affirmative, 'tis convex.*

For by Prop III. Fig. 91. $sn = \ddot{y} - \frac{\dot{z}\dot{z}}{y}$, which is affirmative if n lie above s, but negative when 'tis below s. And therefore $y\ddot{y} - \dot{z}\dot{z}$ will be affirmative in the former caſe; that is, when the curve is convex; and negative in the latter, or when the curve is concave.

Cor. *Hence if a curve is referred to a fixed point, to know whether it is concave or convex; put the equation of the curve into fluxions, and making $\dot{x}$ conſtant, find the value of $y\ddot{y} - \dot{z}\dot{z}$; then if it be negative the curve is concave towards the center; if affirmative, 'tis convex.*

Ex. 1.

Let the equation of the curve be $y\dot{x} = \frac{\dot{z}\sqrt{ay}}{2}$.

Then $\dot{x}^2 = \frac{\dot{z}^2 \times ay}{4yy} = \frac{a\dot{z}^2}{4y} = \frac{a}{4y} \times \overline{\dot{x}^2 + \dot{y}^2}$, whence $\dot{y}^2 = \frac{4y - a}{a} \times \dot{x}^2$, and $\dot{y} = \frac{\dot{x}}{\sqrt{a}}\sqrt{4y - a}$, and $\ddot{y} = \frac{\dot{x}}{\sqrt{a}} \times \frac{2\dot{y}}{\sqrt{4y - a}}$. Alſo $\dot{z} = \frac{2y\dot{x}}{\sqrt{ay}}$, and $\dot{z}\dot{z} = \frac{4yy\dot{x}\dot{x}}{ay} = \frac{4y\dot{x}\dot{x}}{a}$; therefore $y\ddot{y} - \dot{z}\dot{z} = \frac{2yy\dot{y}\dot{x}}{\sqrt{a}\sqrt{4y - a}}$

$-\frac{4y\dot{x}\dot{x}}{a} = \frac{2y}{\sqrt{4y-a}} \times \frac{\dot{x}^2}{a}\sqrt{4y-a} - \frac{4y\dot{x}^2}{a} =$ Fig. 91.
$\frac{2y\dot{x}^2}{a} - \frac{4y\dot{x}\dot{x}}{a} = -\frac{2y\dot{x}\dot{x}}{a}$, which being negative the curve is concave towards the center.

Ex. 2.

Let ABM be a ſpiral, FBR the generating circle, radius CR $= r$, arch RD $= v$, CM $= y$, and let the nature of the curve be denoted by the equation $vy = aa$. By the ſimilar ſectors DC*d*, MC*o*, $r : \dot{v} :: y : \dot{x}$, whence $y\dot{v} = r\dot{x}$. And putting the equation of the curve into fluxions, we have $v\dot{y} + y\dot{v} = 0$, and $v\dot{y} = -y\dot{v} = -r\dot{x}$; then $\dot{y} = -\frac{r\dot{x}}{v} = \frac{-ry\dot{x}}{aa}$, and $\ddot{y} = -\frac{r\dot{y}\dot{x}}{aa} = -\frac{r\dot{x}}{aa} \times \frac{-ry\dot{x}}{aa} = \frac{rry\dot{x}\dot{x}}{a^4}$, and $y\ddot{y} = \frac{r^2y^2\dot{x}^2}{a^4}$. Alſo $\dot{z}\dot{z} = \dot{x}^2 + \dot{y}^2 = \dot{x}^2 + \frac{rryy\dot{x}^2}{a^4} = \frac{a^4\dot{x}^2 + rryy\dot{x}^2}{a^4}$. Therefore $y\ddot{y} - \dot{z}\dot{z} = \frac{rryy\dot{x}\dot{x}}{a^4} - \frac{a^4 + r^2y^2}{a^4}\dot{x}^2 = -\dot{x}^2$; which being negative the curve is concave to the center. 95.

Ex. 3.

But if the equation of the curve be $vy^3 = a^4$, and $r = a$, then $3vy^2\dot{y} + y^3\dot{v} = 0$; that is (becauſe $y\dot{v} = a\dot{x}$) $\frac{3a^4\dot{y}}{y} + ay^2\dot{x} = 0$, whence $\dot{y} = -\frac{y^3\dot{x}}{3a^5}$, and $\ddot{y} = -\frac{y^2\dot{x}\dot{y}}{a^3}$, and $y\ddot{y} = -\frac{y^3\dot{x}\dot{y}}{a^3} = -\frac{y^3\dot{x}}{a^3} \times -\frac{y^3\dot{x}}{3a^3} = \frac{y^6\dot{x}^2}{3a^6}$. Alſo $\dot{z}^2 = \dot{x}^2 + \dot{y}^2 = \frac{9a^6\dot{x}^2}{9a^6} + \frac{y^6\dot{x}^2}{ya^5}$. Therefore $y\ddot{y} - \dot{z}\dot{z} = \frac{y^6\dot{x}^2}{3a^6} - \frac{9a^6 + y^6}{9a^6}\dot{x}^2 =$

Fig. 95. $\frac{3y^6 - 9a^6 - y^6}{9a^6}\dot{x}^2 = \frac{2y^6 - 9a^6}{9a^6}\dot{x}^2$. Therefore if $2y^6$ be leſſer than $9a^6$, the curve is concave; but if greater, convex towards the center.

PROP. XIV.

101. *Let* AMC *be a curve,* AP $= x$, PM $= y$, AM $= z$, M*m* $= \dot{z}$, M*n* $= x$, *mn* $= \dot{y}$, *and let* $\dot{y}$ *be given; and if a and b be given quantities, and* $a\dot{x} = b\dot{z}$ *be the nature of the curve, then will* $a\dot{z} - b\dot{x}$ *be a maximum or minimum.*

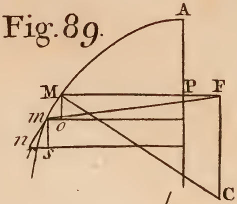
For taking the fluxion of $a\dot{z} - b\dot{x}$, we have $a\ddot{z} - b\ddot{x} = 0$, and $a\ddot{z} = b\ddot{x}$. But $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$; therefore $2\dot{z}\ddot{z} = 2\dot{x}\ddot{x}$, and $\dot{z}\ddot{z} = \dot{x}\ddot{x}$; whence by dividing, $\ddot{z} = \frac{\dot{x}\ddot{x}}{\dot{z}}$, and $a\ddot{z} = \frac{a\dot{x}\ddot{x}}{\dot{z}}$; therefore $\frac{a\dot{x}\ddot{x}}{\dot{z}} = b\ddot{x}$, and $a\dot{x} = b\dot{z}$. Therefore on the contrary when $a\dot{x} = b\dot{z}$, then $a\dot{z} - b\dot{x}$ will be a maximum or minimum.

Cor. *If* $a\dot{x} = b\dot{z}$ *be the nature of a curve, then the* Fl: $a\dot{z}$ — Fl: $b\dot{x}$ *will be a maximum or minimum.*

For $a\dot{z} - b\dot{x}$ is a maximum or minimum, and therefore the ſum of all the $a\dot{z}$ — the ſum of all the $b\dot{x}$ will be a maximum or minimum; that is, fluent of $a\dot{z}$ — fluent of $b\dot{x}$ will be a maximum or minimum.

PROP.

Fig. 89.



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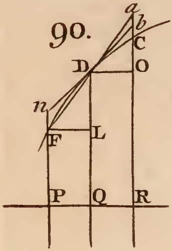

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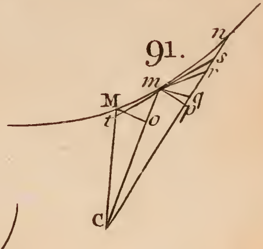

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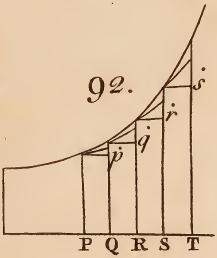

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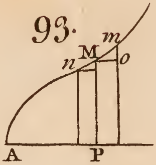

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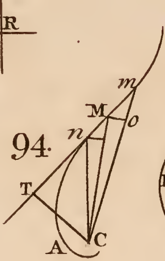

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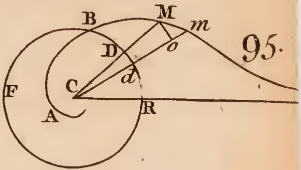

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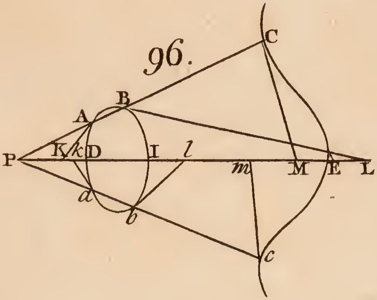

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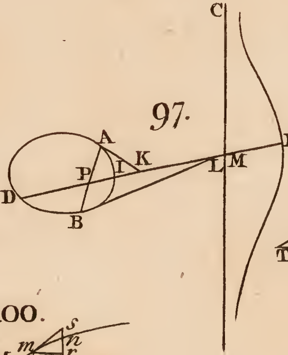

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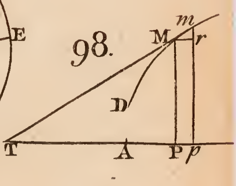

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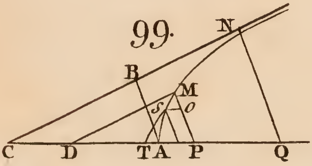

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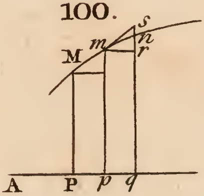

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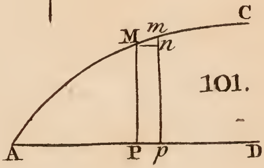

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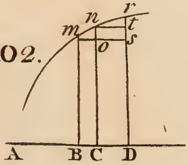

Pl.VIII *the En*

Fig. 102.

PROP XV.

If Bm, Dr, *be two ordinates of a curve, infinitely near; and let the ordinate* Cn *be drawn, ſo that the differences of the ordinates on, tr, may be equal. Then put mo* $= \dot x$, *os or nt* $= \dot v$, *on or tr* $= \dot y$; *then if* $a\dot x + b\dot v = d$, *a given quantity; and* $p \times \overline{\dot x^2 + \dot y^2}^{\,n} + q \times \overline{\dot v^2 + \dot y^2}^{\,n} =$ *a maximum or minimum; then* $pb\dot x \times \overline{\dot x^2 + \dot y^2}^{\,n-1} = qa\dot v \times \overline{\dot v^2 + \dot y^2}^{\,n-1}$ *will denote the nature of the curve.*

For putting the two given equations into fluxions, $a\ddot x + b\ddot v = 0$, and $\ddot v = -\frac{a\ddot x}{b}$. Alſo $2pn\dot x\ddot x \times \overline{\dot x^2 + \dot y^2}^{\,n-1} + 2qn\dot v\ddot v \times \overline{\dot v\dot v + \dot y\dot y}^{\,n-1} = 0$, and $2pn\dot x\ddot x \times \overline{\dot x^2 + \dot y^2}^{\,n-1} - \frac{2qna\ddot x\dot v}{b} \times \overline{\dot v^2 + \dot y^2}^{\,n-1} = 0$, and tranſpoſing and dividing $bp\dot x \times \overline{\dot x\dot x + \dot y\dot y}^{\,n-1} = aq\dot v \times \overline{\dot v\dot v + \dot y\dot y}^{\,n-1}$.

Cor. *Hence* $\frac{p}{a} \times mo \times mn^{2n-2} = \frac{q}{b} \times nt \times nr^{2n-2}$.

For $\dot x\dot x + \dot y\dot y = mn^2$, and $\dot v^2 + \dot y^2 = nr$, whence $pb\dot x \times mn^{2n-2} = qa\dot v \times nr^{2n-2}$, and $\frac{p\dot x}{a} \times mn^{2n-2} = \frac{q\dot v}{b} \times nr^{2n-2}$.

Many more properties of curves might be obtained by the method of fluxions; but I forbear purſuing them any further; ſince what have been here delivered, are ſufficient for an introduction.

FINIS.

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ERRATA.

For the AR. INFINITES.

b ſignifies reckon from the bottom.

page	line	read
3	3*b*	$s = \overline{\overline{n+1}\,.\,a}^{\,m}$
7	13	all the *x*es
12	11	$c = -A + 2b + C =$
44	0*b*	FINIS. *Dele* THE.

CONIC SECTIONS.

Where *b* follows, reckon from the bottom.

page	line	read
18	6*b*	XIII. Cor. 1.)
34	5*b*	OD × DN : :
56	7	*lines el,*
60	2	ib. Prop. III.)
131	4	multiplying by 4.

CURVE LINES.

89	7	whence *ya*
99	2	$= y\dot{x} \times \frac{\dot{y}\ddot{y}}{\dot{z}}$

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